Numerical Modelling in Geosciences

Lecture 10

Advection

Advection equation

$$Eulerian\ \ point: \frac{\partial A}{\partial t} = -\vec{v}\nabla A$$

$$3D: \frac{\partial A}{\partial t} = -v_x\left(\frac{\partial A}{\partial x}\right) - v_y\left(\frac{\partial A}{\partial y}\right) - v_z\left(\frac{\partial A}{\partial z}\right)$$

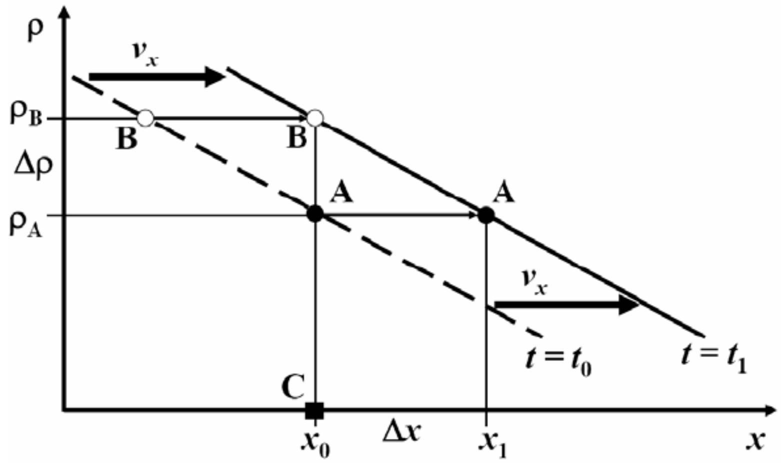
$$Lagrangian\ \ point: \frac{Dx_i}{Dt} = v_i$$

$$3D: \begin{cases} \dfrac{Dx}{Dt} = v_x \\ \dfrac{Dy}{Dt} = v_y \\ \dfrac{Dz}{Dt} = v_z \end{cases}$$

$$Eulerian\ \ point: \frac{\partial \rho}{\partial t} = -\vec{v}\nabla \rho$$

$$1D: \frac{\partial \rho}{\partial t} = -v_x\left(\frac{\partial \rho}{\partial x}\right)$$

$$Lagrangian\ \ point: \frac{D\rho}{Dt} = 0$$



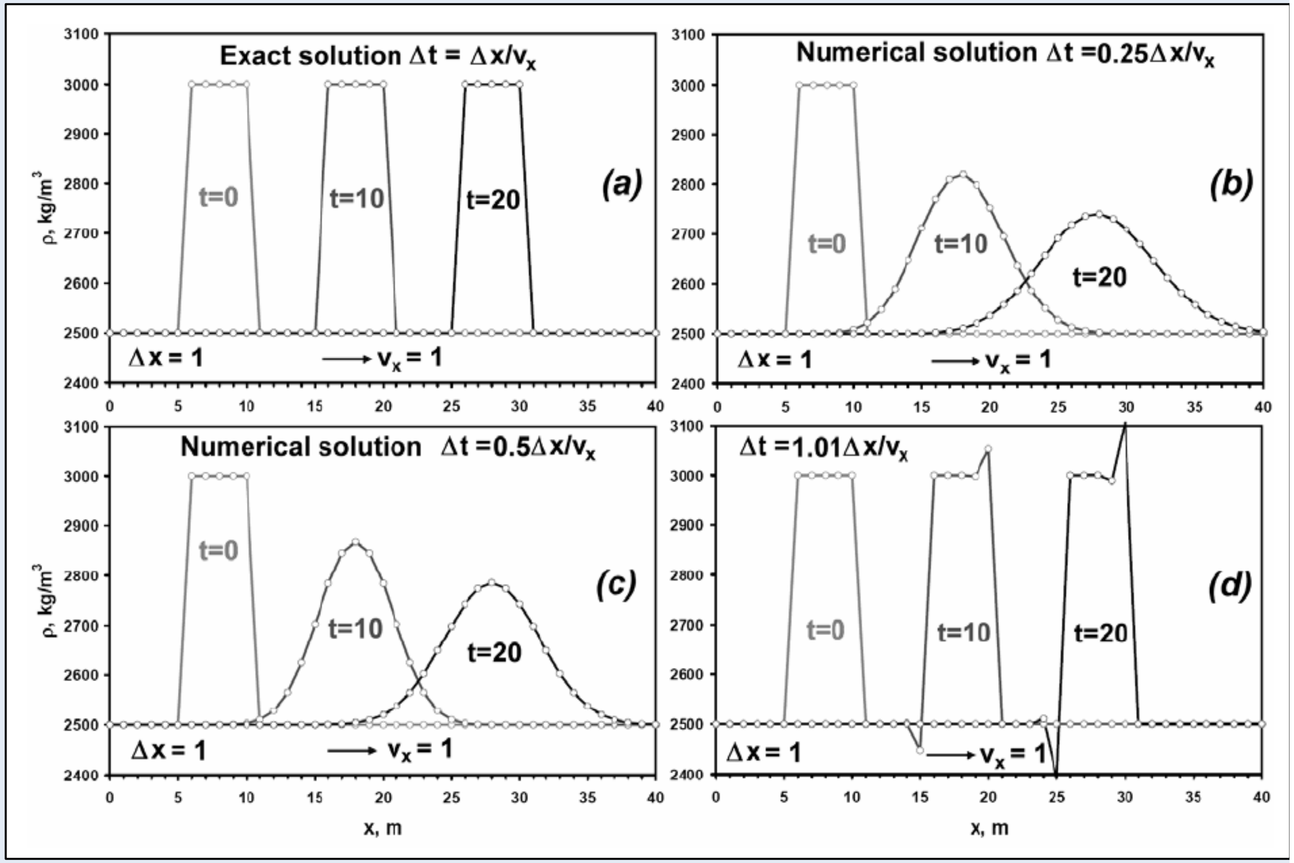
Eulerian advection schemes

Eulerian advection scheme are affected by strong numerical diffusion of physical properties during advection of sharp gradients.
Numerical diffusion depends on the number of iterations.
Advection is limited by the **Courant factor**: **$dt \le dx/2/v_{max}$**

$$1D\ case: \frac{\partial \rho}{\partial t} = -\vec{v}\left(\frac{\partial \rho}{\partial x}\right)$$

Upwind differences

$$\rho_i^{t+\Delta t} = \rho_i^t - v_x \Delta t \frac{\rho_i^t - \rho_{i-1}^t}{\Delta x}$$



Eulerian advection schemes

$$1D\ case: \frac{\partial \rho}{\partial t} = -\vec{v}\left(\frac{\partial \rho}{\partial x}\right)$$

Upwind FD

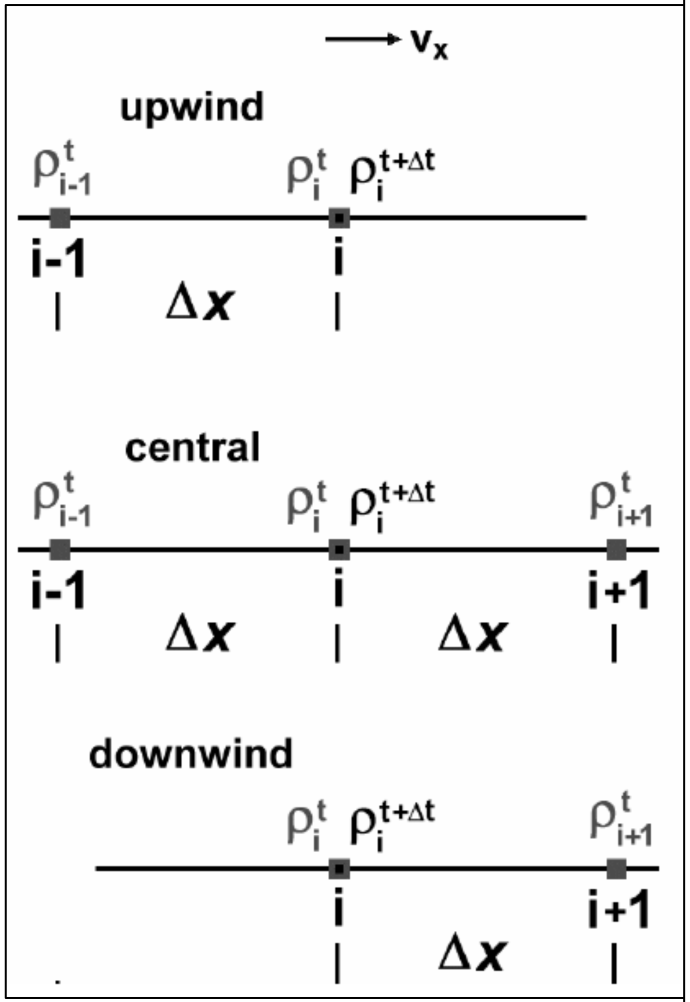
$$\rho_i^{t+\Delta t} = \rho_i^t - v_x \Delta t \frac{\rho_i^t - \rho_{i-1}^t}{\Delta x}$$

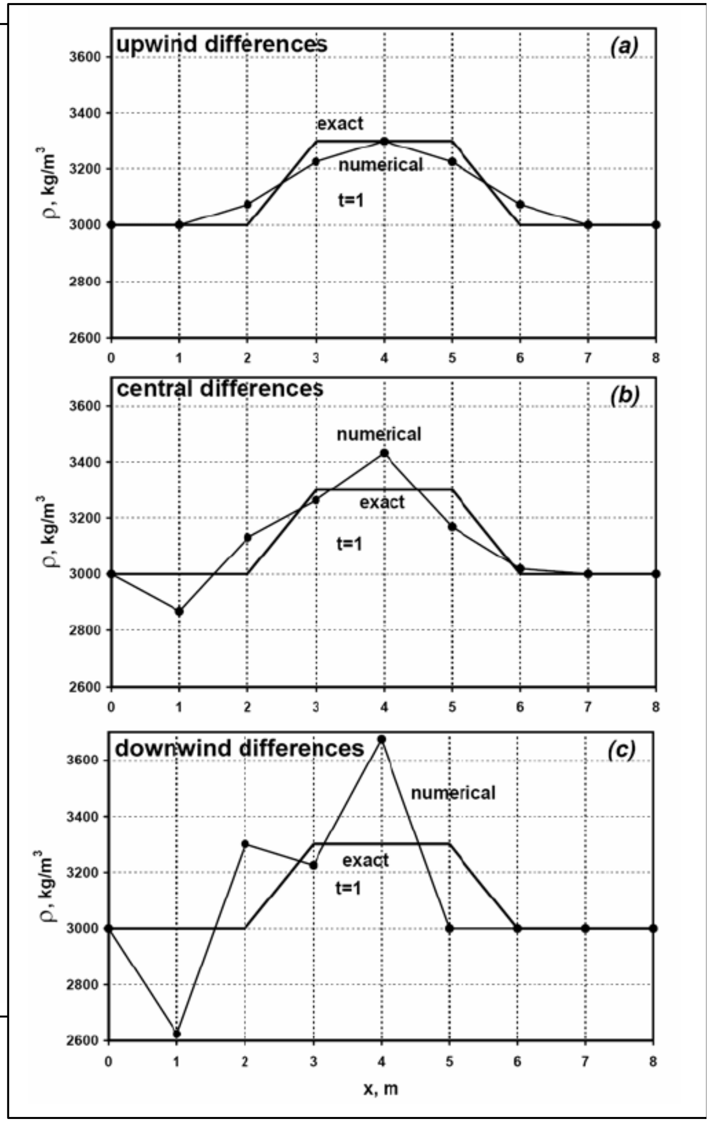
Central FD

$$\rho_i^{t+\Delta t} = \rho_i^t - v_x \Delta t \frac{\rho_{i+1}^t - \rho_{i-1}^t}{2\Delta x}$$

Downwind FD

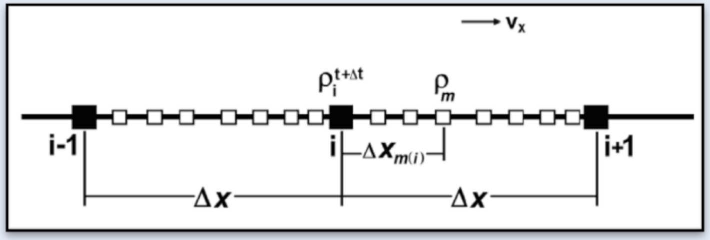
$$\rho_i^{t+\Delta t} = \rho_i^t - v_x \Delta t \frac{\rho_{i+1}^t - \rho_i^t}{\Delta x}$$





Marker-in-cell technique

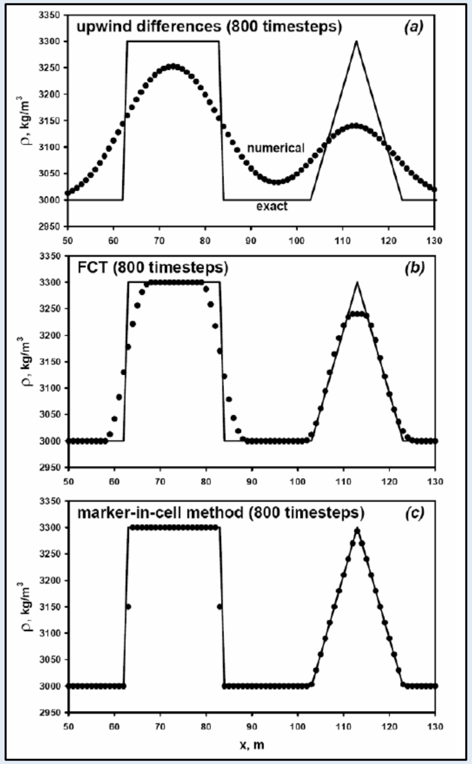
When large advection or deformation of material best solution is to use a mixed Eulerian-Lagrangian technique:
Material properties are assigned to a large amount of Lagrangian points (markers, tracers or particles) that are advected according to the velocity field. The advected properties are then interpolated from the displaced Lagrangian points to the Eulerian grid by using various interpolation schemes such as linear interpolation:



1D linear interpolation

$$\rho_i^{t+\Delta t} = \frac{\sum_m \rho_m w_{m(i)}}{\sum_m w_{m(i)}}$$

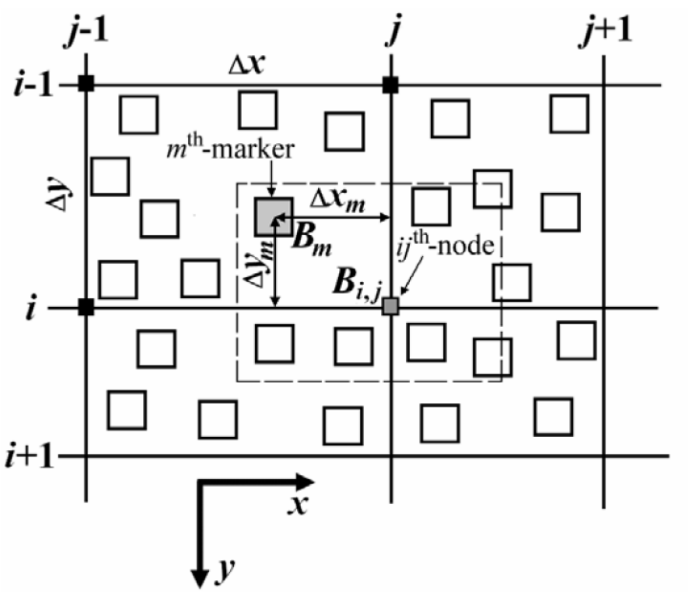
$$w_{m(i)} = 1 - \frac{\Delta x_{m(i)}}{\Delta x}$$



Marker-in-cell technique

2D interpolation: bi-linear interpolation: **LET'S PRACTICE!**

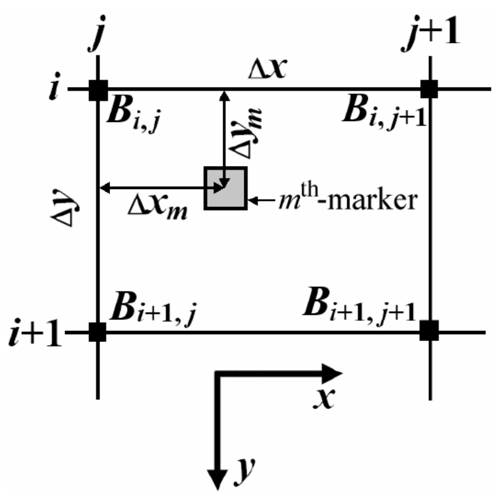
From marker to nodes



$$B_{i,j} = \frac{\sum_m B_m w_{m(i,j)}}{\sum_m w_{m(i,j)}},$$

$$w_{m(i,j)} = \left(1 - \frac{\Delta x_m}{\Delta x}\right) \times \left(1 - \frac{\Delta y_m}{\Delta y}\right),$$

From nodes to markers

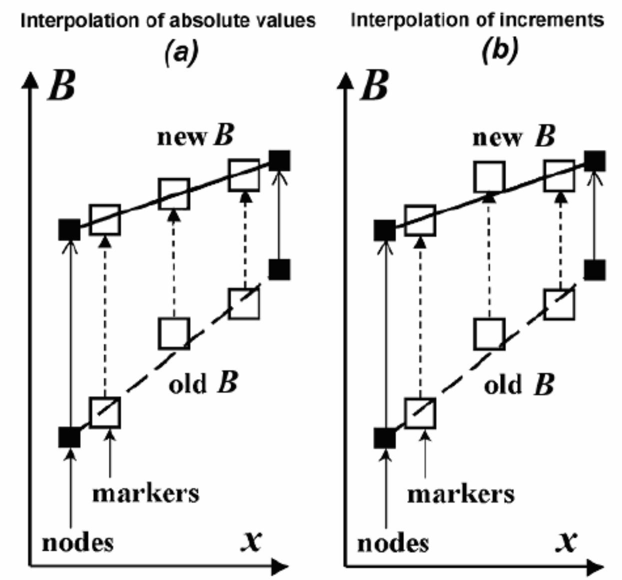


$$B_m = B_{i,j}\left(1 - \frac{\Delta x_m}{\Delta x}\right)\left(1 - \frac{\Delta y_m}{\Delta y}\right) + B_{i,j+1}\frac{\Delta x_m}{\Delta x}\left(1 - \frac{\Delta y_m}{\Delta y}\right) + $$
$$+ B_{i+1,j}\left(1 - \frac{\Delta x_m}{\Delta x}\right)\frac{\Delta y_m}{\Delta y} + B_{i+1,j+1}\frac{\Delta x_m \Delta y_m}{\Delta x \Delta y},$$

Marker-in-cell technique

2D interpolation: bi-linear interpolation of increments rather than absolute values minimize numerical diffusion due to interpolation (e.g., for temperature):

From nodes to markers



$$B_m^{t+\Delta t} = B_m^t + \left(B_{i,j}^{t+\Delta t} - B_{i,j}^t\right)\left(1-\frac{\Delta x_m}{\Delta x}\right)\left(1-\frac{\Delta y_m}{\Delta y}\right) + \left(B_{i,j+1}^{t+\Delta t} - B_{i,j+1}^t\right)\frac{\Delta x_m}{\Delta x}\left(1-\frac{\Delta y_m}{\Delta y}\right) +$$

$$+\left(B_{i+1,j}^{t+\Delta t} - B_{i+1,j}^t\right)\left(1-\frac{\Delta x_m}{\Delta x}\right)\frac{\Delta y_m}{\Delta y} + \left(B_{i+1,j+1}^{t+\Delta t} - B_{i+1,j+1}^t\right)\frac{\Delta x_m \Delta y_m}{\Delta x \Delta y}$$

Lagrangian point advection

If constant velocity field:

$$x_A^{t+\Delta t} = x_A^t + v_{xA}\Delta t$$
$$y_A^{t+\Delta t} = y_A^t + v_{yA}\Delta t$$
$$z_A^{t+\Delta t} = z_A^t + v_{zA}\Delta t$$

Otherwise:

$$x_A^{t+\Delta t} = x_A^t + v_x^{eff}\Delta t$$
$$y_A^{t+\Delta t} = y_A^t + v_y^{eff}\Delta t$$
$$z_A^{t+\Delta t} = z_A^t + v_z^{eff}\Delta t$$

where the effective velocity components are calculated with the Runge – Kutta advection scheme

Temporal discretization

Euler method:1 (initial) function evalution,
error 1 order of magnitude less than approximation

Function y(t) is position h = dt f(t,y) = velocity

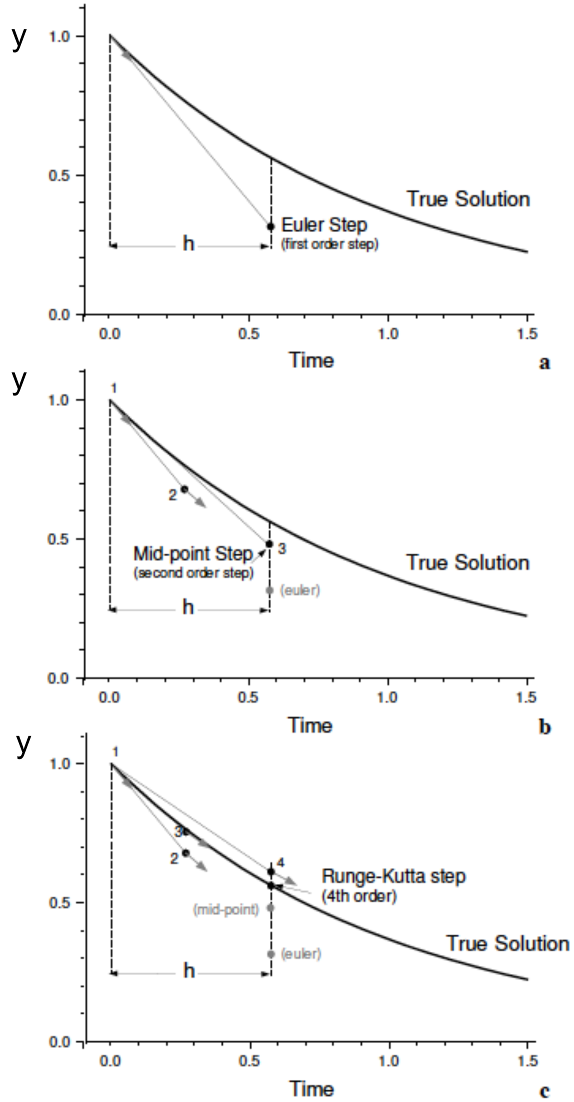
$$y(t_0 + h) = y_0 + hf(t_0, y_0) + O(h^2)$$

Mid-point method: 2 function evalutions,
error 2 orders of magnitude less than approximation

$$\begin{aligned} k_1 &= hf(t_n, y_n) \\ k_2 &= hf(t_n + h/2, y_n + k_1/2) \\ y_{n+1} &= y_n + k_2 + O(h^3) \end{aligned}$$

Runge-Kutta method: 4 function evalutions,
error 4 orders of magnitude less than approximation

$$\begin{aligned} k_1 &= hf(t_n, y_n) \\ k_2 &= hf(t_n + h/2, y_n + k_1/2) \\ k_3 &= hf(t_n + h/2, y_n + k_2/2) \\ k_4 &= hf(t_n + h, y_n + k_3) \\ y_{n+1} &= y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(h^5) \end{aligned}$$



4th-order Runge-Kutta scheme

Given a function y_n whose derivatives depends on time and space, its value y_{n+1} at t + Δt is given by current value y_n plus a the weighted average of 4 increments :

$$y^{t+\Delta t} = y^t + \frac{1}{6}\Delta t \cdot \left(k_A + 2k_B + 2k_C + k_D\right)$$

$$where:$$

$$k_A = f\left(t^t, y^t\right),$$

$$k_B = f\left(t^t + \frac{1}{2}\Delta t, y^t + \frac{\Delta t}{2}k_A\right),$$

$$k_C = f\left(t^t + \frac{1}{2}\Delta t, y^t + \frac{\Delta t}{2}k_B\right),$$

$$k_D = f\left(t^t + \Delta t, y^t + \Delta t k_C\right)$$

4th order accuracy in space and time

4th-order Runge-Kutta scheme

For material displacement:

$$x_A^{t+\Delta t} = x_A^t + \frac{1}{6}\Delta t\left(k_A + 2k_B + 2k_C + k_D\right) = x_A^t + \frac{1}{6}\Delta t\left(v_{xA} + 2v_{xB} + 2v_{xC} + v_{xD}\right)$$

$$where:$$

$$k_A = f\left(t_t, x_A^t\right) = v_{xA},$$

$$k_B = f\left(t_t, x_A^t + \frac{\Delta t}{2}k_A\right) = f\left(t_t, x_B^t\right) = v_{xB},$$

$$k_C = f\left(t_t, x_A^t + \frac{\Delta t}{2}k_B\right) = f\left(t_t, x_C^t\right) = v_{xC},$$

$$k_D = f\left(t_t, x_A^t + \Delta t k_C\right) = f\left(t_t, x_D^t\right) = v_{xD}$$

4th order accuracy in space,1st order in time

4th-order Runge-Kutta scheme

$v_{xA}, v_{xB}, v_{xC}, v_{xD}, v_{yA}$…. are interpolated from the surrounding nodes at positions x_A, x_B, x_C, x_D

$$x_A^{t+\Delta t} = x_A^t + v_x^{eff}\Delta t\,,$$

$$y_A^{t+\Delta t} = y_A^t + v_y^{eff}\Delta t\,,$$

$$z_A^{t+\Delta t} = z_A^t + v_z^{eff}\Delta t\,,$$

$$x_B = x_A^t + v_{xA}\frac{\Delta t}{2}, \quad y_B = y_A^t + v_{yA}\frac{\Delta t}{2}, \quad z_B = z_A^t + v_{zA}\frac{\Delta t}{2},$$

$$x_C = x_A^t + v_{xB}\frac{\Delta t}{2}, \quad y_C = y_A^t + v_{yB}\frac{\Delta t}{2}, \quad z_C = z_A^t + v_{zB}\frac{\Delta t}{2},$$

$$x_D = x_A^t + v_{xC}\Delta t\,, \quad y_D = y_A^t + v_{yC}\Delta t\,, \quad z_D = z_A^t + v_{zC}\Delta t\,.$$

$$v_x^{eff} = \frac{1}{6}\left(v_{xA} + 2v_{xB} + 2v_{xC} + v_{xD}\right),$$

$$v_y^{eff} = \frac{1}{6}\left(v_{yA} + 2v_{yB} + 2v_{yC} + v_{yD}\right),$$

$$v_z^{eff} = \frac{1}{6}\left(v_{zA} + 2v_{zB} + 2v_{zC} + v_{zD}\right),$$

4th-order Runge-Kutta scheme

For tracking finite strain:

$$F_{ij}^{t+1} = F_{ij}^{t} + \frac{1}{6}\Delta t\left(A_{ij1} + 2A_{ij2} + 2A_{ij3} + A_{ij4}\right)$$

$$where:$$

$$A_{ij1} = L_{ij}^{t} F_{ij}^{t} \;,$$

$$A_{ij2} = L_{ij}^{t}\left(F_{ij}^{t} + \frac{\Delta t}{2} A_{ij1}\right),$$

$$A_{ij3} = L_{ij}^{t}\left(F_{ij}^{t} + \frac{\Delta t}{2} A_{ij2}\right),$$

$$A_{ij4} = L_{ij}^{t}\left(F_{ij}^{t} + \Delta t\, A_{ij3}\right).$$

Exercise

1) Define physical properties on Lagrangian particles
2) Interpolate physical properties (density, viscosity) to the Eulerian nodes
3) Solve momentum and continuity equations
4) Define timestep with Courant criterion
5) Advect particles by interpolating the velocity field from Eulerian nodes
6) Repeat from point 2

Homework

Read chapter 8 of textbook:

Gerya, T. *Introduction to numerical geodynamic modelling*. Cambridge University Press, 345 pp. (2010)