Numerical Modelling in Geosciences

Lecture 2

Equation of mass conservation

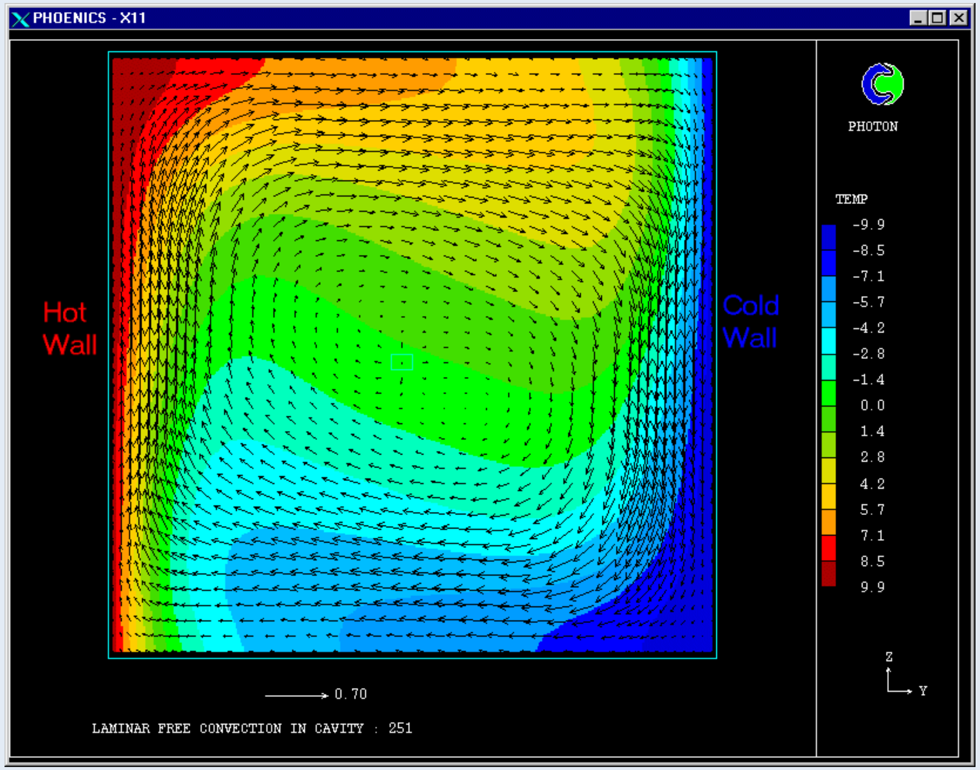
Continuum mechanics

Geological media are considered as continuous mass rather than as discrete particles. In reality the matter is made of atoms, and so it is not continuous. However, on length scales much greater than inter-atomic distances, models assuming an object as continuous are highly accurate.

In continuum mechanics, the medium completely fills the space it occupies: no voids!

For example, cracks and pores are considered to be filled by fluids/air.

Displacement in a continuous medium implies conservation of mass described by the continuity equation.



Continuity equation

Eulerian formulation

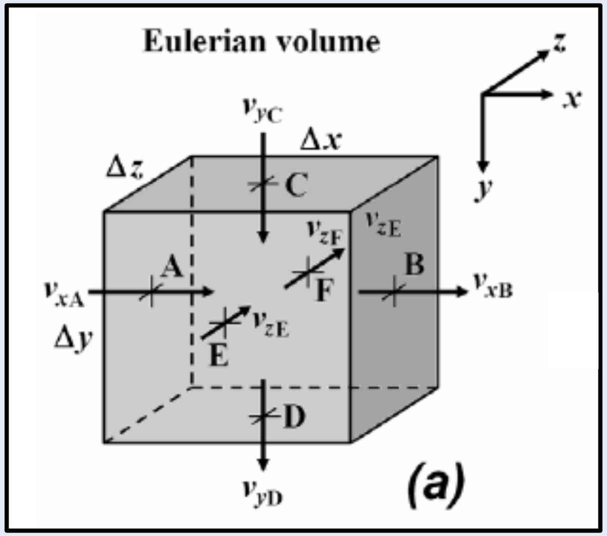
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{v} \right) = 0$$

Lagrangian formulation

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0$$

Eulerian continuity equation

An Eulerian point is immobile, not connected with any material point.
Time variation of density due to outward/inward of mass flux. Volume is constant.
The rate of density change is equal to minus the divergence of the mass flux.



$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\vec{q}_{mass} \right) = \frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{v} \right) = 0$$

$$Mass\ \ flux\ \ \vec{q}_{mass} = \rho \vec{v}\ \left(kg / m^2 / s \right)$$

$$Eulerian\ \ time\ \ derivative \Rightarrow \frac{\partial f}{\partial t}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \left(\rho \vec{v} \right)$$

Let's derive it…

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}\frac{m}{V} = \frac{1}{V}\frac{\partial m}{\partial t} \approx \frac{1}{V}\frac{\Delta m}{\Delta t} = \frac{1}{\Delta x \Delta y \Delta z}\frac{m^{t+\Delta t} - m^{t}}{\Delta t} \ldots$$

Lagrangian continuity equation

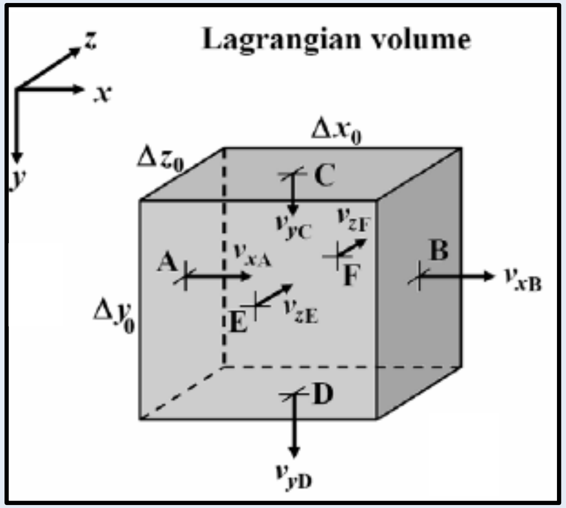
A Lagrangian point is connected with a given material point and is moving with this point.
Variation of density due to variation of volume (expansion/contraction). Mass is constant.
The rate of logarithmic density change is equal to minus the divergence of the velocity vector

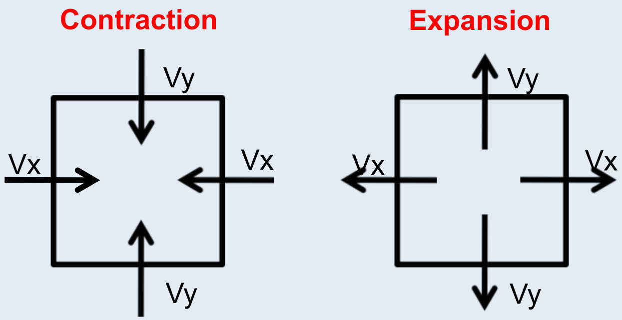
$$\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\vec{v}\right)=\frac{\partial\rho}{\partial t}+\vec{v}\cdot\nabla\rho+\rho\nabla\cdot\vec{v}=0$$

$$\textit{Lagrangian time derivative}\Rightarrow\frac{Df}{Dt}=\frac{\partial f}{\partial t}+\vec{v}\cdot\nabla f$$

$$\frac{D\rho}{Dt}+\rho\nabla\cdot\vec{v}=0$$

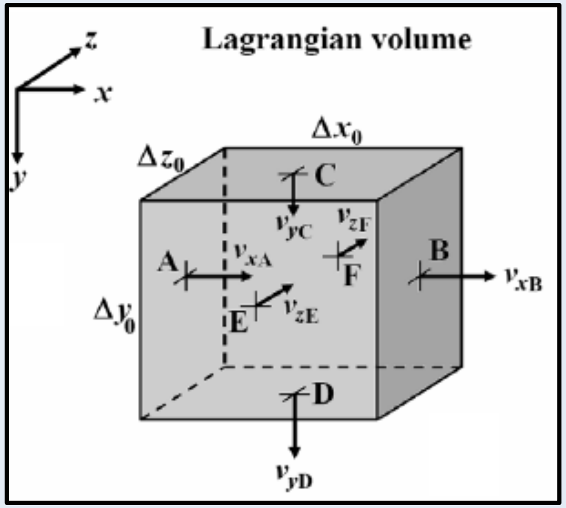
$$\frac{D\ln\rho}{Dt}=-\nabla\cdot\vec{v}$$





$$\frac{D\ln\rho}{Dt}>0\rightarrow\nabla\cdot\vec{v}<0 \qquad \frac{D\ln\rho}{Dt}<0\rightarrow\nabla\cdot\vec{v}>0$$

Derive Lagrangian continuity equation



1) $$\frac{D\rho}{Dt} = \frac{D}{Dt}\frac{m}{V} = m\frac{D}{Dt}\frac{1}{V} \approx m\frac{\Delta\, {}^{1}\!/_{V}}{\Delta t} = \frac{m}{\Delta t}\left(\frac{1}{\Delta x_1 \Delta y_1 \Delta z_1} - \frac{1}{\Delta x_0 \Delta y_0 \Delta z_0}\right)$$

2) $$\frac{\Delta\rho}{\Delta t} = \rho_0 \frac{1 - \left(1+\Delta t\frac{\Delta v_x}{\Delta x_0}\right)\left(1+\Delta t\frac{\Delta v_y}{\Delta y_0}\right)\left(1+\Delta t\frac{\Delta v_z}{\Delta z_0}\right)}{\Delta t\left(1+\Delta t\frac{\Delta v_x}{\Delta x_0}\right)\left(1+\Delta t\frac{\Delta v_y}{\Delta y_0}\right)\left(1+\Delta t\frac{\Delta v_z}{\Delta z_0}\right)}$$

for Δt → 0, this term goes to 0

3) $$\frac{\Delta\rho}{\Delta t} = \rho_0 \frac{-\frac{\Delta v_x}{\Delta x_0} - \frac{\Delta v_y}{\Delta y_0} - \frac{\Delta v_z}{\Delta z_0} - \Delta t\left(\frac{\Delta v_x}{\Delta x_0}\frac{\Delta v_y}{\Delta y_0} + \frac{\Delta v_x}{\Delta x_0}\frac{\Delta v_z}{\Delta z_0} + \frac{\Delta v_y}{\Delta y_0}\frac{\Delta v_z}{\Delta z_0} + \Delta t\frac{\Delta v_x}{\Delta x_0}\frac{\Delta v_y}{\Delta y_0}\frac{\Delta v_z}{\Delta z_0}\right)}{\left(1+\Delta t\frac{\Delta v_x}{\Delta x_0}\right)\left(1+\Delta t\frac{\Delta v_y}{\Delta y_0}\right)\left(1+\Delta t\frac{\Delta v_z}{\Delta z_0}\right)}$$

for Δt → 0, this term goes to 1

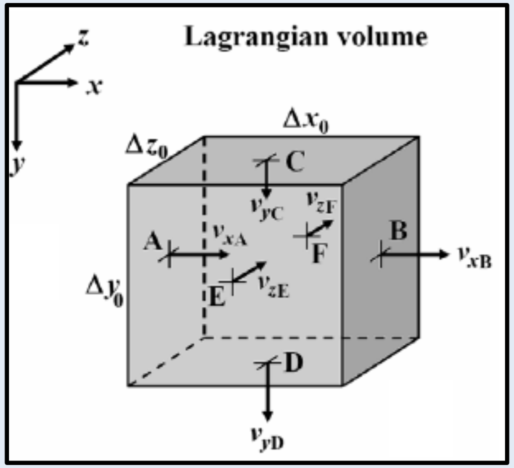
4) $$\frac{D\rho}{Dt} + \rho\frac{\partial v_x}{\partial x} + \rho\frac{\partial v_y}{\partial y} + \rho\frac{\partial v_z}{\partial z} = 0$$

Incompressibility

We can derive the continuity equation for incompressible media from the Lagrangian formulation.

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{v} = 0$$

$$\nabla \cdot \vec{v} = -\frac{1}{\rho}\frac{D\rho}{Dt} = -\frac{D\ln(\rho)}{Dt}$$



The material point density should not change otherwise we would expect compaction/dilation (i.e., volume variation of the material point).

$$\boxed{\nabla \cdot \vec{v} = 0}$$

The incompressible continuity equation is valid for both Eulerian and Lagrangian formulations

We will use this type of continuity equation!

Difference between Eulerian and Lagrangian formulations

EULERIAN

A Eulerian point is immobile, not connected with any material point.
Variation of density due to variation of mass flux (outward/inward). Volume is constant

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \left(\rho \vec{v}\right) = -\rho \nabla \cdot \vec{v} - \vec{v} \cdot \nabla \rho$$

$$\frac{\partial x}{\partial t} \quad (\textit{Eulerian time derivative})$$

LAGRANGIAN

A Lagrangian point is connected with a given material point and is moving with this point.
Variation of density due to variation of volume (expansion/contraction). Mass is constant

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \vec{v}$$

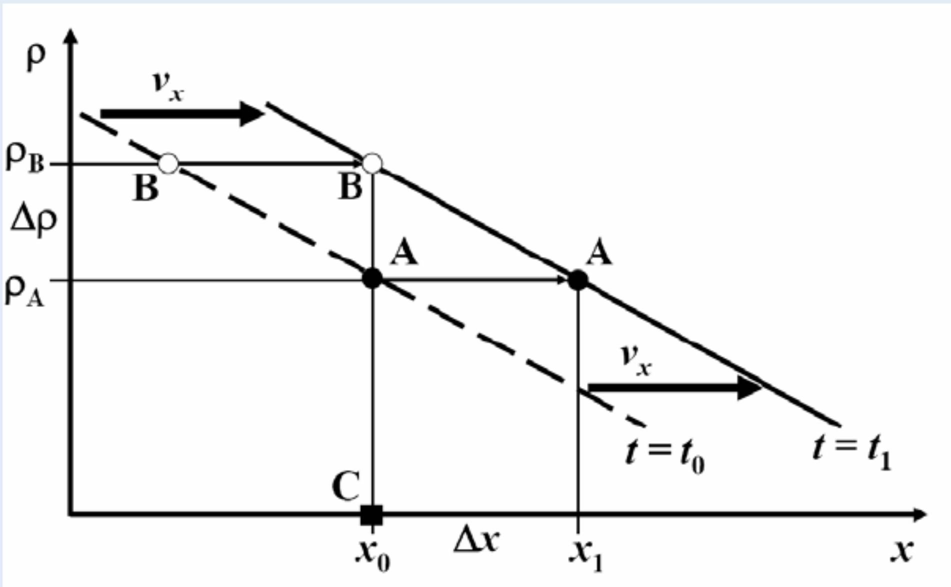
$$\frac{Dx}{Dt} = \frac{\partial x}{\partial t} + \vec{v} \cdot \nabla x \quad (\textit{Lagrangian time derivative})$$

Advective term: rate of change of density for an immobile, Eulerian point due to displacement of a medium with density gradients.

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho$$

If material is incompressible:

$$\frac{D\rho}{Dt} = 0 \underset{\textit{Advective transport equation}}{\Rightarrow} \frac{\partial \rho}{\partial t} = -\vec{v} \cdot \nabla \rho$$



Exercise

Analytical exercise

Exercise 1.1. In a region of the Earth's mantle, the velocity field is given by

$$v_x = 10^{-10} + x \cdot 10^{-13} + y \cdot 10^{-13} + z \cdot 10^{-13}$$

$$v_y = 10^{-10} - x \cdot 10^{-13} + y \cdot 2\times10^{-13} + z \cdot 3\times10^{-13}$$

$$v_z = 10^{-10} - x \cdot 10^{-13} - y \cdot 10^{-13} - z \cdot 2\times10^{-13}$$

The mantle density field in the same region is given by

$$\rho = 3300 + x \cdot 0.001 - y \cdot 0.002 + z \cdot 0.001.$$

Calculate ρ, $\mathrm{div}(\vec{v})$, $\frac{\partial \rho}{\partial t}$ and $\frac{D\rho}{Dt}$ for the point with coordinates x=1000, y=1000, z=1000.

Numerical Modelling in Geosciences

Practice 2

Equation of mass conservation

Exercise

Exercise 1.2. Write a MATLAB code for computing and visualising a 2D velocity field and its divergence. Model design: an area of the mantle (1000x1500 km) is convecting with one central upwelling in the middle of the model box and two downwellings at the sides. The velocity field is given by the following equations

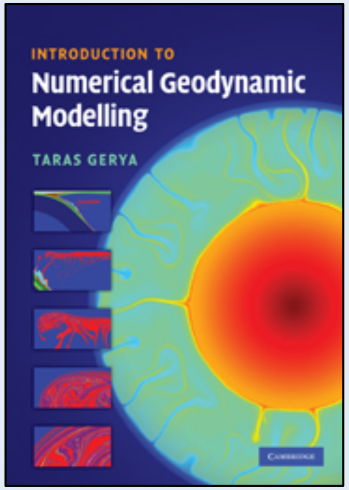
$$v_x = -v_{x0} \sin\left(2\pi \frac{x}{W}\right)\cos\left(\pi \frac{y}{H}\right),$$

$$v_y = v_{y0} \cos\left(2\pi \frac{x}{W}\right)\sin\left(\pi \frac{y}{H}\right),$$

where x and y are respectively horizontal and vertical coordinates inside the box in m; W=1000000 m and H=1500000 m are width and height of the model, respectively (i.e., 1000x1500 km); $v_{x0}=10^{-9}$ m/s and $v_{y0}=10^{-9}$ m/s are scaling values for respectively horizontal and vertical velocity components (10^{-9} m/s ≈ 3 cm/year). Compute (analytically) v_x, v_y, $\frac{\partial v_x}{\partial x}$, $\frac{\partial v_y}{\partial y}$ and $div(\overline{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}$ on a 2D grid of points (e.g. 31x31) which are regularly distributed inside the model and visualise these parameters separately as colormaps (*pcolor*) in order to see how they are distributed relative to each other. Visualise the velocity as an arrow field (*quiver*).

Homework

Read the chapter 1 of textbook:



Gerya, T. *Introduction to numerical geodynamic modelling*. Cambridge University Press, 345 pp. (2010)

Finish Exercise 1.2