Numerical Modelling in Geosciences

Lecture 4

Poisson’s equation

Poisson's equation

$$\Delta\varphi = f$$

$$\nabla^2\varphi = f$$

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = f$$

where φ and f are two functions. It is used in many field of physics.
We will use it to model:

- the gravitational potential
- chemical/thermal diffusion.
- pipe flow

Poisson's equation

Analytical solution in 1D:

$$\frac{\partial^2 \varphi}{\partial x^2} = f$$

$$\varphi = f\frac{x^2}{2} + C_1 x + C_2$$

$$if \;\; x = 0 \Rightarrow \varphi = \varphi_0 \Rightarrow C_2 = \varphi_0$$

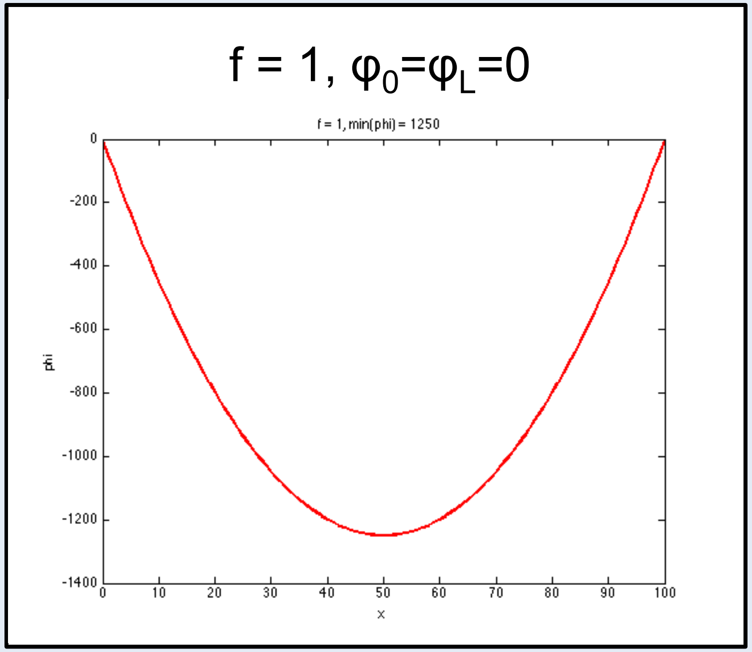
$$if \;\; x = L \Rightarrow \varphi = \varphi_L \Rightarrow C_1 = -f\frac{L}{2} + \frac{\varphi_L - \varphi_0}{L}$$

$$\varphi = \frac{f}{2}\left(x^2 - Lx\right) + \frac{\varphi_L - \varphi_0}{L} + \varphi_0$$

$$if \;\; \varphi_L = \varphi_0 \Rightarrow \varphi = \frac{f}{2}\left(x^2 - Lx\right) + \varphi_0$$

$$if \;\; \varphi_0 = 0 \Rightarrow \varphi = \frac{f}{2}\left(x^2 - Lx\right)$$

$$\min(\varphi) = -\frac{f}{8}L^2$$



Poisson's equation for gravity

It can be used to measure the gravitational potential Φ (J/kg, or gravitational potential energy per unit mass) due to the density distribution ρ(x,y,z) of a self-gravitating medium:

$$\Delta\Phi = 4\pi G\rho(x,y,z)$$

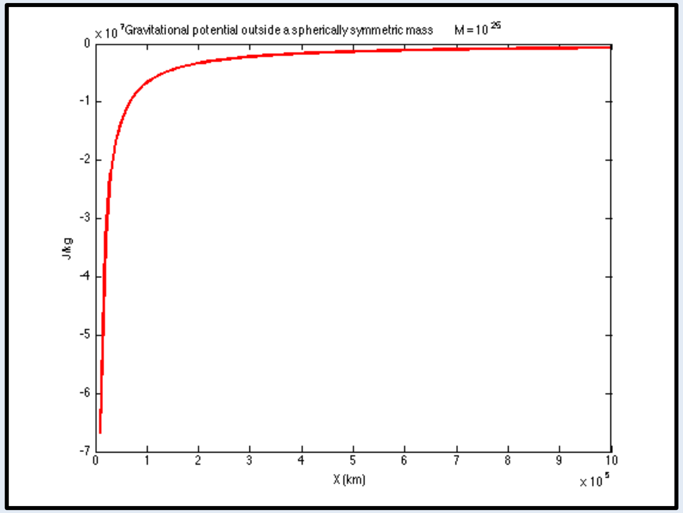
$$\nabla^2\Phi = 4\pi G\rho(x,y,z)$$

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 4\pi G\rho(x,y,z)$$

where $G = 6.672 \times 10^{-11}$ N m^2/kg^2

The gravitational potential is the negative of the work done to bring a unit mass from infinity to a position inside the gravitational field produced by M. Indeed, the mass M act as a gravitational potential sink.

Outside a spherically symmetric mass M:



$$\Phi = -\frac{GM}{r}$$

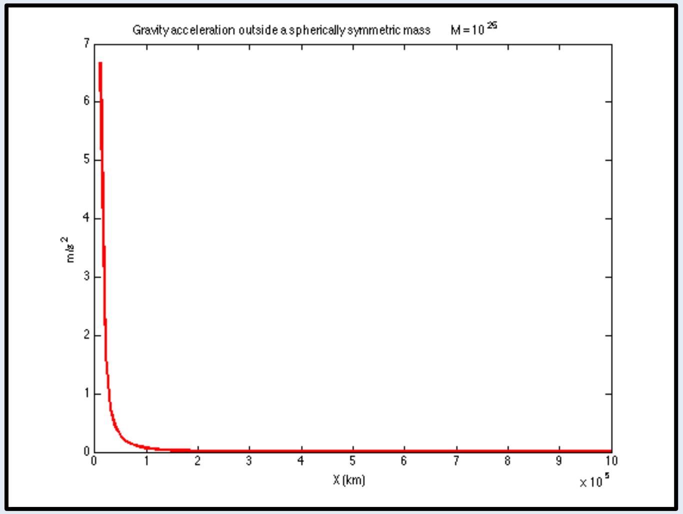
At infinity Φ → 0

Absolute values of Φ at the Earth's surface due to the:

Earth	Sun	Milky way
60 MJ/Kg	900 MJ/kg	≥ 130 GJ/kg

Poisson's equation for gravity

Local derivatives of Φ by the spatial coordinates give the components of the gravitational acceleration vector: $\frac{f}{m} = \vec{g} = -\frac{GM}{r^2}$



$$\frac{\partial \Phi}{\partial x} = -g_x$$

$$\frac{\partial \Phi}{\partial y} = -g_y$$

$$\frac{\partial \Phi}{\partial z} = -g_z$$

The variation of Φ is equal to minus the force needed to move a unit mass over *dr*, that is, Φ increases against the force of attraction (g can be seen as force per unit mass). Please, show that:

$$\Delta \Phi = -\nabla \cdot \vec{g}$$

Poisson's equation for gravity

Derivation for a point source of mass *dm* with radius dr and constant density *ρ*:

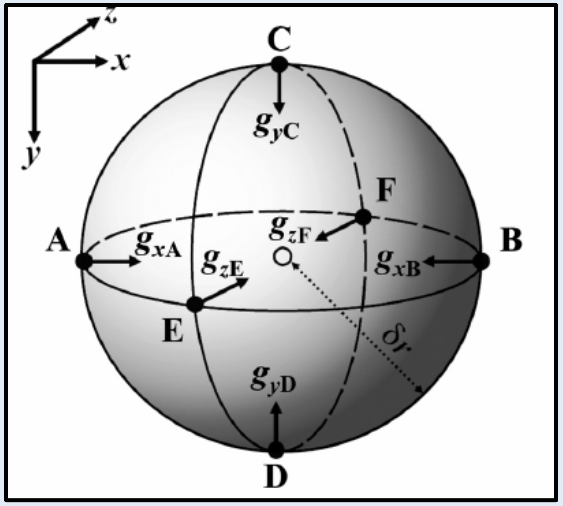
$$\mathrm{div}(\vec{g}) = \frac{g_{xB} - g_{xA}}{2\delta r} + \frac{g_{yD} - g_{yC}}{2\delta r} + \frac{g_{zF} - g_{zE}}{2\delta r}$$

$$g_{xA} = g_{yC} = g_{zE} = G\frac{\delta m}{\delta r^2}$$

$$g_{xB} = g_{yD} = g_{zF} = -G\frac{\delta m}{\delta r^2}$$

$$\rho = \frac{\delta m}{V} = \frac{3\delta m}{4\pi\delta r^3}$$

$$\Delta\Phi = -\mathrm{div}(\vec{g}) = 3G\frac{\delta m}{\delta r^3} = 4\pi G\rho$$



Homework

Read pp. 30-34 of textbook:

Gerya, T. *Introduction to numerical geodynamic modelling*. Cambridge University Press, 345 pp. (2010)