Numerical Modelling in Geosciences

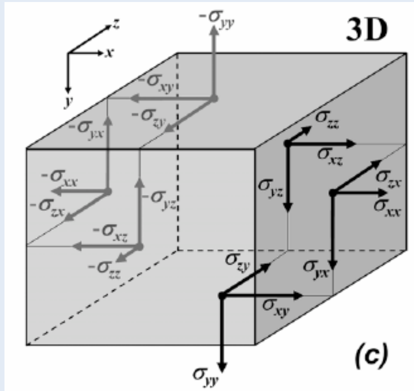
Lecture 6

Deformation

Tensor

Second-rank tensor → stress (σ), strain (ε), strain rate ($\dot{\varepsilon}$)

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$



Invariants (quantities independent of the coordinate system):

- First invariant → trace: $tr\left(\sigma_{ij}\right) = \left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)$

- Second invariant → magnitude:

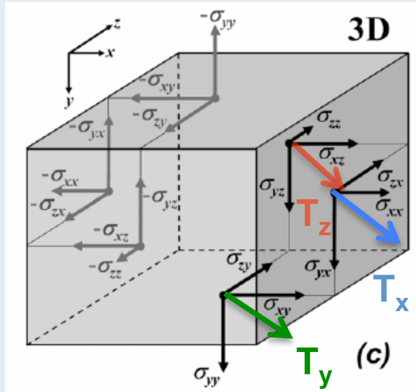
$$\left\|\sigma\right\| = \sqrt{\tfrac{1}{2}\sigma_{ij}^2} = \sqrt{\tfrac{1}{2}\left(\sigma_{xx}^2 + \sigma_{xy}^2 + \sigma_{xz}^2 + \sigma_{yx}^2 + \sigma_{yy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2 + \sigma_{zy}^2 + \sigma_{zz}^2\right)}$$

- Third invariant → determinant

Stress

In geodynamics compressional (extensional) stress are negative (positive). Pressure is positive under compression. Stress is measured in Pa = N/m^2. The stress tensor contains the components of the tractions acting on the element surfaces. The first index indicate the direction of stress, the second the normal to the stressed surface

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$



Pressure is equal to the mean normal stress:

$$2D: \quad P = -\frac{tr(\sigma)}{2} = -\frac{\sigma_{kk}}{2} = -\frac{\sigma_{xx} + \sigma_{yy}}{2}$$

$$3D: \quad P = -\frac{tr(\sigma)}{3} = -\frac{\sigma_{kk}}{3} = -\frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}$$

In absence of internal angolar momentum, the tensor is symmetric:

$$\sigma_{xy} = \sigma_{yx}$$

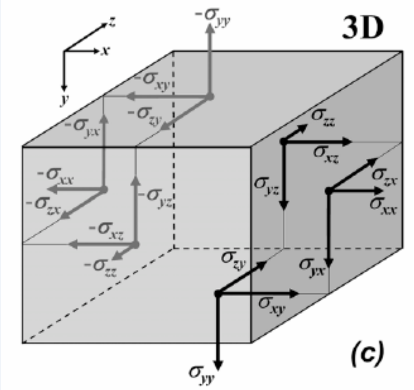
$$\sigma_{xz} = \sigma_{zx}$$

$$\sigma_{zy} = \sigma_{yz}$$

Deviatoric Stress

Stress can be divided into a deviatoric and an isotropic components. The deviatoric components produce flow, the isotropic components (i.e., pressure) compaction/dilation.

$$\sigma_{ij} = \sigma'_{ij} + \delta_{ij}\frac{\sigma_{kk}}{3} = \sigma'_{ij} - P\delta_{ij} \quad where \quad \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$



$$3D: \sigma'_{ij} = \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} & \sigma'_{xz} \\ \sigma'_{yx} & \sigma'_{yy} & \sigma'_{yz} \\ \sigma'_{zx} & \sigma'_{zy} & \sigma'_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx}+P & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy}+P & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz}+P \end{bmatrix}$$

$$Normal\ deviatoric\ stress: \sigma'_{ii} = \sigma_{ii} + P;$$

$$Shear\ deviatoric\ stress: \sigma'_{ij} = \sigma'_{ji} = \sigma_{ij} = \sigma_{ji}$$

$$Please,\ show\ that\ tr\left(\sigma'_{ij}\right) = 0$$

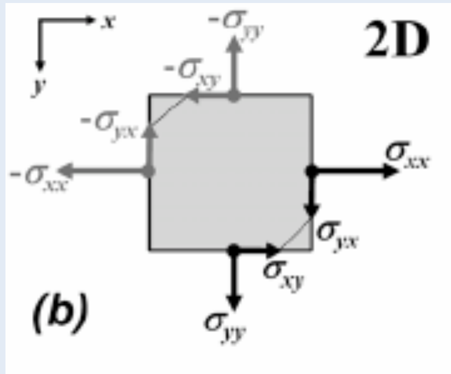
$$Second\ invariant\ of\ deviatoric\ stress\ tensor:$$

$$\sigma'_{II} = \sqrt{1/2\,\sigma'^2_{ij}} = \sqrt{1/2\left(\sigma'^2_{xx} + \sigma'^2_{yy} + \sigma'^2_{zz}\right) + \sigma'^2_{xy} + \sigma'^2_{xz} + \sigma'^2_{yz}}$$

Deviatoric Stress

Stress can be divided into a deviatoric and an isotropic components. The deviatoric components produce flow, the isotropic component (i.e., pressure) compaction/dilation

$$2D:\ \sigma'_{ij} = \begin{bmatrix} \sigma'_{xx} & \sigma'_{xy} \\ \sigma'_{yx} & \sigma'_{yy} \end{bmatrix} = \begin{bmatrix} \sigma_{xx}+P & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy}+P \end{bmatrix}$$



$$Please,\ show\ that\ in\ 2D \Rightarrow \sigma'_{xx} = -\sigma'_{yy}$$

$$Second\ \ invariant\ \ of\ \ deviatoric\ \ stress\ \ tensor:$$

$$\sigma'_{II} = \sqrt{\tfrac{1}{2}\sigma'^2_{ij}} = \sqrt{\sigma'^2_{xx} + \sigma'^2_{xy}}$$

Tectonic pressure

If a medium is at rest, then no deviatoric stresses exist and the total pressure is equal to the lithostatic pressure.

$$P_{TOT} = P_{LITH} = P_0 + g\int_0^z \rho(z)\,dz$$

When deformation is applied, the total pressure is equal to the lithostatic pressure + the tectonic pressure

$$P_{TOT} = P_{LITH} + P_{TECT}$$

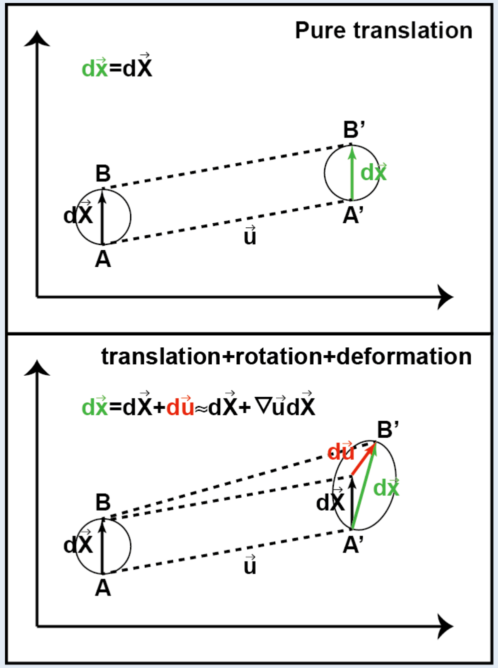
$$P_{TECT} = -\frac{\sigma_{kk}}{3} - P_{LITH}$$

Exercise:
Calculate the lithostatic, total and tectonic pressures, and the total and deviatoric stress tensors at 10 km depth in the crust (density = 2500 kg/m^3, g_z=10 m/s^2), when a compressional stress of 150 MPa is applied along the x direction.

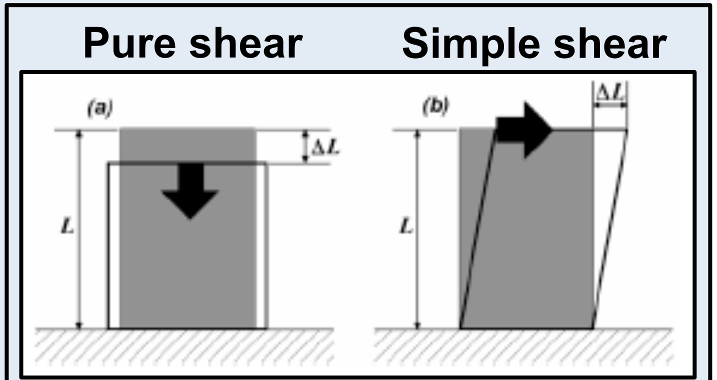
Displacement, its gradient and velocity

$$Displacement: \vec{u} = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

$$Rate\ of\ displacement: \frac{D\vec{u}}{Dt} = \vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$



$$Displacement\ \ gradient: \nabla\vec{u} = \frac{\partial u_i}{\partial j} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix}$$



$$\frac{\partial u_y}{\partial y} \approx \frac{\Delta L}{L} \qquad \frac{\partial u_x}{\partial y} \approx \frac{\Delta L}{L}$$

Displacement gradient, strain, rotation

Displacement gradient (adimensional)= Strain (symmetric) + Rotation (anti-symmetric)

$$\nabla\vec{u} = \frac{\partial u_i}{\partial j} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

$$\nabla\vec{u} = \frac{1}{2}\left(\nabla\vec{u} + \nabla\vec{u}^T\right) + \frac{1}{2}\left(\nabla\vec{u} - \nabla\vec{u}^T\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial j} - \frac{\partial u_j}{\partial i}\right)$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i}\right) \quad (Strain)$$

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial j} - \frac{\partial u_j}{\partial i}\right) \quad (Rotation)$$

$$\nabla\vec{u} = \varepsilon + \omega$$

$$\nabla\vec{u} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} - \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) & 0 \end{bmatrix}$$

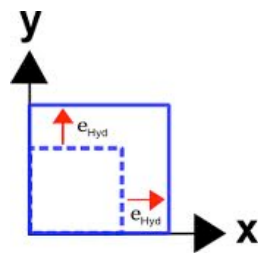
Strain

Indicates the amount of deformation and is adimensional

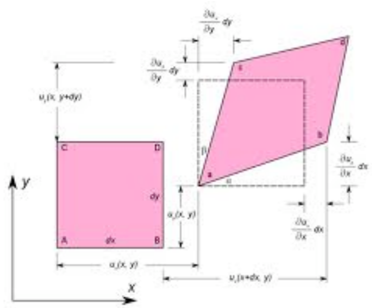
$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i}\right) = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

$$Volumetric\ \ strain = \Delta V = tr\left(\varepsilon_{ij}\right) = \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$If\ \ incompressible \Rightarrow \Delta V = tr\left(\varepsilon_{ij}\right) = \nabla \cdot \vec{u} = 0$$



$$Deviatoric\ \ strain: \varepsilon'_{ij} = \varepsilon_{ij} - \frac{1}{3}\delta_{ij}\varepsilon_{kk}$$



Velocity gradient, strain rate, vorticity

Velocity gradient (s^{-1})= Strain rate (symmetric) + Vorticity (anti-symmetric)

$$\nabla\vec{v} = \frac{\partial v_i}{\partial j} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

$$\nabla\vec{v} = \frac{1}{2}\left(\nabla\vec{v} + \nabla\vec{v}^T\right) + \frac{1}{2}\left(\nabla\vec{v} - \nabla\vec{v}^T\right) = \frac{1}{2}\left(\frac{\partial v_i}{\partial j} + \frac{\partial v_j}{\partial i}\right) + \frac{1}{2}\left(\frac{\partial v_i}{\partial j} - \frac{\partial v_j}{\partial i}\right)$$

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial j} + \frac{\partial v_j}{\partial i}\right) \quad (Strain\ rate)$$

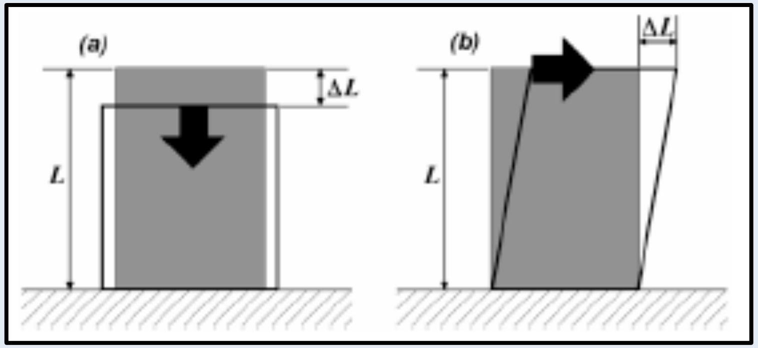
$$\dot{\omega}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial j} - \frac{\partial v_j}{\partial i}\right) \quad (Vorticity,\ rotation\ rate)$$

$$\nabla\vec{v} = \dot{\varepsilon} + \dot{\omega}$$

$$\nabla\vec{v} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) & \frac{\partial v_y}{\partial y} & \frac{1}{2}\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) & \frac{\partial v_z}{\partial z} \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial v_y}{\partial z} - \frac{\partial v_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial v_z}{\partial x} - \frac{\partial v_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) & 0 \end{bmatrix}$$

End-member flows

Pure shear	Simple shear



$$Pure\ Shear: \nabla\vec{v} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \dot{\varepsilon} + \dot{\omega} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$Simple\ Shear: \nabla\vec{v} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \dot{\varepsilon} + \dot{\omega} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Strain rate

The strain rate tensor can be divided into a deviatoric and isotropic components. Dimension is (s^{-1}). The trace of the strain rate tensor gives the rate of volume change

$$\dot{\varepsilon}_{ij} = \begin{bmatrix} \dot{\varepsilon}_{xx} & \dot{\varepsilon}_{xy} & \dot{\varepsilon}_{xz} \\ \dot{\varepsilon}_{yx} & \dot{\varepsilon}_{yy} & \dot{\varepsilon}_{yz} \\ \dot{\varepsilon}_{zx} & \dot{\varepsilon}_{zy} & \dot{\varepsilon}_{zz} \end{bmatrix} = \frac{1}{2}\left(\frac{\partial v_i}{\partial j} + \frac{\partial v_j}{\partial i}\right) = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right) & \frac{\partial v_y}{\partial y} & \frac{1}{2}\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right) & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}'_{ij} + \frac{1}{3}\dot{\varepsilon}_{kk}\delta_{ij}$$

$$tr\left(\dot{\varepsilon}_{ij}\right) = \dot{\varepsilon}_{kk} = \dot{\varepsilon}_{11} + \dot{\varepsilon}_{22} + \dot{\varepsilon}_{33} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \nabla \cdot \vec{v} = \Delta\dot{V}$$

$$If\ \ incompressible \Rightarrow \Delta\dot{V} = tr\left(\dot{\varepsilon}_{ij}\right) = \nabla \cdot \vec{v} = 0$$

$$tr\left(\dot{\varepsilon}'_{ij}\right) = 0$$

$$Second\ \ invariant\ \ of\ \ deviatoric\ \ strain\ \ rate\ \ tensor: \dot{\varepsilon}'_{II} = \sqrt{\tfrac{1}{2}\dot{\varepsilon}'^{2}_{ij}}$$

$$\begin{aligned} \dot{\varepsilon}_{xy} &= \dot{\varepsilon}_{yx} = \dot{\varepsilon}'_{xy} = \dot{\varepsilon}'_{yx} \\ \dot{\varepsilon}_{xz} &= \dot{\varepsilon}_{zx} = \dot{\varepsilon}'_{xz} = \dot{\varepsilon}'_{zx} \\ \dot{\varepsilon}_{zy} &= \dot{\varepsilon}_{yz} = \dot{\varepsilon}'_{zy} = \dot{\varepsilon}'_{yz} \end{aligned}$$

Displacement vs deformation gradient

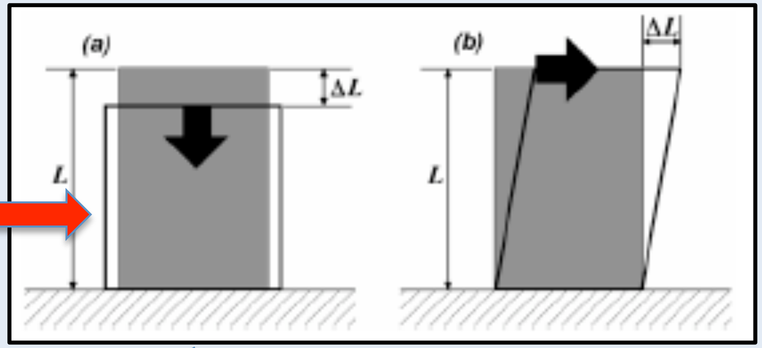
$$\nabla\vec{u} = \frac{\partial u_i}{\partial j} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{bmatrix} \Rightarrow \frac{\partial u_y}{\partial y} = \frac{\Delta L}{L}$$

Displacement gradient

Length variation over the initial length

Pure shear

Simple shear

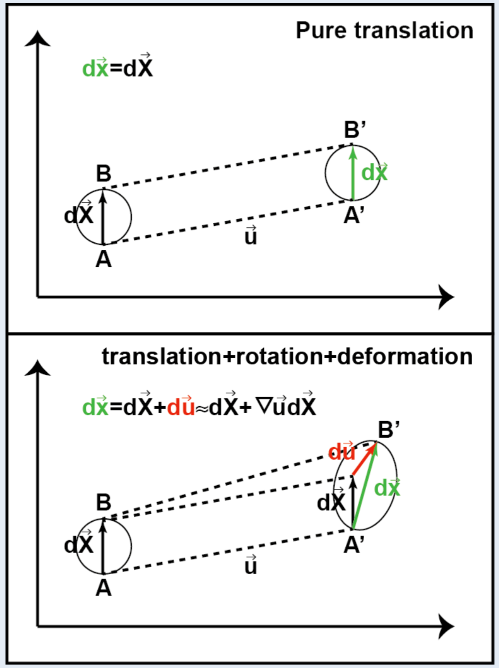


$$F_{ij} = I + \nabla\vec{u} = \delta_{ij} + \frac{\partial u_i}{\partial j} = \begin{bmatrix} 1+\frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & 1+\frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & 1+\frac{\partial u_z}{\partial z} \end{bmatrix} \Rightarrow$$

Deformation gradient

$$\Rightarrow F_{yy} = \frac{L+\Delta L}{L} = 1 + \frac{\partial u_y}{\partial y}$$

Final length over the initial length



Time derivative of deformation gradient

We can track the history of deformation for a given Lagrangian particle by computing the time derivative of F, and successively by isolating the amount of deformation from the amount of rotation.
For homogeneous, steady-state flow:

$$\frac{\partial F}{\partial t} = \nabla\vec{v} \cdot F \Rightarrow F^{t+\Delta t} = F^{t} + \nabla\vec{v} \cdot F \cdot \Delta t$$

Initial deformation gradient

$$F_{ij}^{t=0} = \delta_{ij} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Left\ \ stretch\ \ tensor : B = F \cdot F^{T}$$

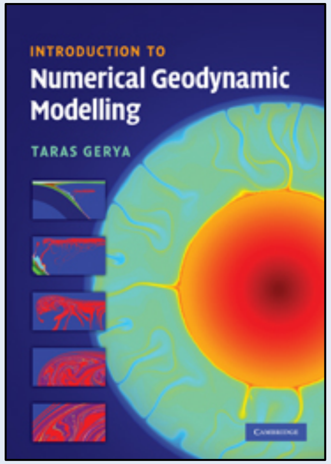
Eigenvalues and eigenvectors of B give, respectively, the square of the magnitude and orientation of the principal stretch axes.

Time to practice…

Homework

Read chapter 4 of textbook:

Gerya, T. *Introduction to numerical geodynamic modelling*. Cambridge University Press, 345 pp. (2010)



Practice with code we have built to track the deformation history