Numerical Modelling in Geosciences

Lecture 8

Conservation of momentum

Conservation of momentum

Newton's second law of motion

$$\vec{f} = m\vec{a} = m\frac{D\vec{v}}{Dt}$$

$$m\vec{v} = momentum$$

$$\vec{f} = \sum_{i=1}^{n} \vec{f}_i^{\,surface} + \sum_{j=1}^{m} \vec{f}_j^{\,volume}$$

Conservation of momentum along x:

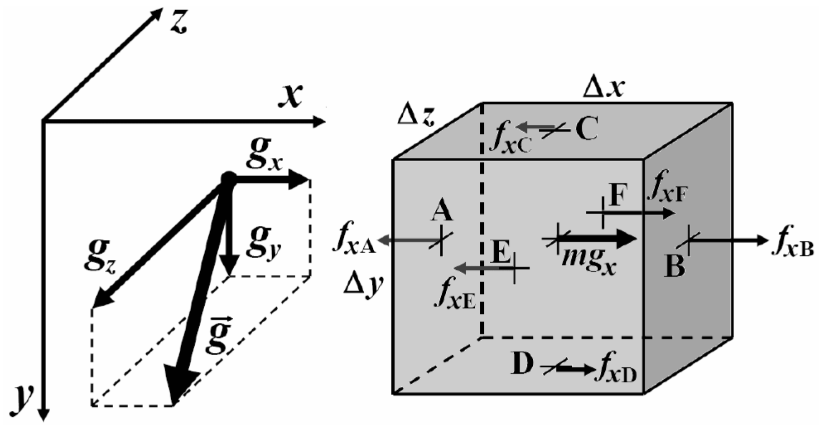
$$f_x = m\frac{Dv_x}{Dt}$$

$$f_x = f_{xA} + f_{xB} + f_{xC} + f_{xD} + f_{xE} + f_{xF} + mg_x$$

$$\Downarrow$$

$$\frac{\partial \sigma_{xj}}{\partial x_j} + \rho g_x = \rho\frac{Dv_x}{Dt}$$

Force balance in a Lagrangian volume along x



Navier-Stokes equation

$$x: \frac{\partial \sigma_{xj}}{\partial x_j} + \rho g_x = \rho \frac{Dv_x}{Dt}$$

$$y: \frac{\partial \sigma_{yj}}{\partial x_j} + \rho g_y = \rho \frac{Dv_y}{Dt}$$

$$z: \frac{\partial \sigma_{zj}}{\partial x_j} + \rho g_z = \rho \frac{Dv_z}{Dt}$$

General equation of conservation of momentum:

$$Eulerian\ \ form: \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i = \rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right)$$

$$Lagrangian\ \ form: \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i = \rho \frac{Dv_i}{Dt}$$

$$\frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + \vec{v} \cdot \nabla v_i$$

Navier-Stokes vs Stokes equations

Since for normal components:

$$\frac{\partial \sigma_{ii}}{\partial x_i} = \frac{\partial \sigma'_{ii}}{\partial x_i} - \frac{\partial P}{\partial x_i}$$

then Navier – Stokes equation can be written as:

$$\frac{\partial \sigma'_{ij}}{\partial x_j} - \frac{\partial P}{\partial x_i} + \rho g_i = \rho \frac{Dv_i}{Dt}$$

$$\frac{\partial}{\partial x_j}\left[\eta\left(\frac{\partial v_i}{\partial j} + \frac{\partial v_j}{\partial i}\right)\right] - \frac{\partial P}{\partial x_i} + \rho g_i = \rho \frac{Dv_i}{Dt}$$

In highly viscous flow such as on Earth $\frac{Dv_i}{Dt} = O(10^{-22}\, m/s^2) \Rightarrow$ *it is negligible*

compared to the term with $g = 9.81\ m/s^2$

Navier – Stokes equation then reduces to Stokes equation:

$$\frac{\partial \sigma'_{ij}}{\partial x_j} - \frac{\partial P}{\partial x_i} + \rho g_i = 0$$

Poisson equation for (Newtonian) channel flow

For constant viscosity and incompressible medium:

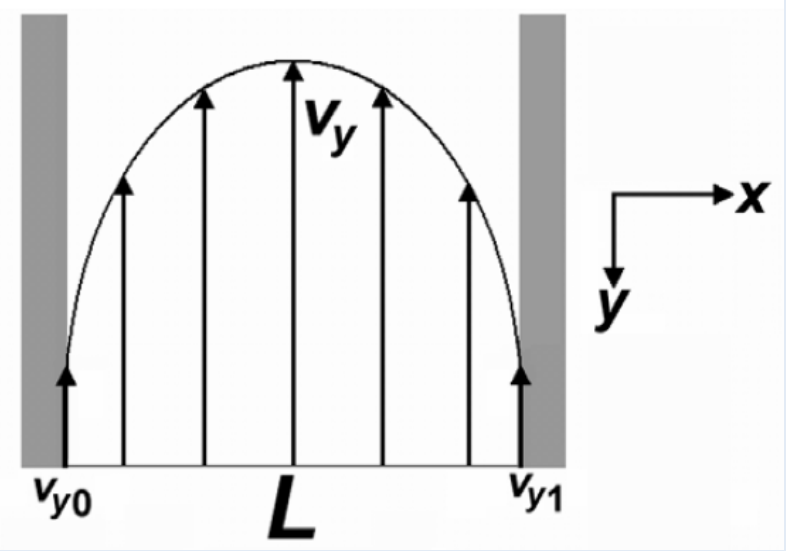
$$\eta \Delta v_i = \frac{\partial P}{\partial x_i} - \rho g_i$$

$$\Downarrow$$

$$\Delta v_i = \frac{1}{\eta}\left(\frac{\partial P}{\partial x_i} - \rho g_i\right)$$

For vertical pipe flow

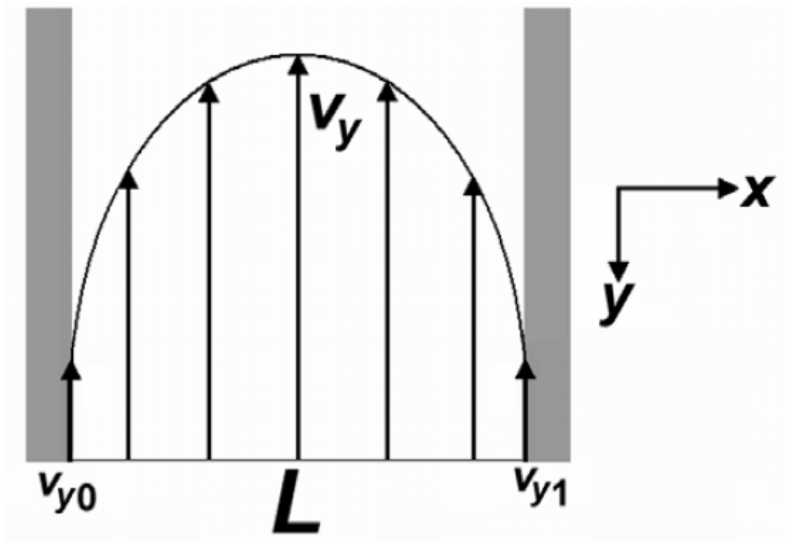
$$\frac{\partial^2 v_y}{\partial x^2} = \frac{1}{\eta}\left(\frac{\partial P}{\partial y} - \rho g_y\right)$$



Please, derive an analytical solution for the previous equation assuming that the right term is constant

non-Newtonian channel flow

$$\frac{\partial \sigma_{xy}}{\partial x} = \frac{\partial P}{\partial y} - \rho g_y$$



$$\sigma_{xy} = \int \left(\frac{\partial \sigma_{xy}}{\partial x} \right) dx = \left(\frac{\partial P}{\partial y} - \rho g_y \right) x + C_1$$

$$\sigma_{xy} = \left(\frac{\dot{\varepsilon}_{xy}}{A} \right)^n$$

$$\frac{\partial v_y}{\partial x} = 2A \left[\left(\frac{\partial P}{\partial y} - \rho g_y \right) x + C_1 \right]^n$$

$$v_y = \frac{2A/(n+1)}{\partial P/\partial y - \rho g_y} \left[\left(\frac{\partial P}{\partial y} - \rho g_y \right) x + C_1 \right]^{n+1} + C_2 \qquad because \int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$if \;\; \sigma_{xy} = 0 \;\; at \;\; x = L/2 \Rightarrow C_1 = -\left(\frac{\partial P}{\partial y} - \rho g_y \right) \frac{L}{2}$$

$$if \;\; v_y = v_{y0} \;\; at \;\; x = 0 \Rightarrow C_2 = v_{y0} - \frac{2A/(n+1)}{\partial P/\partial y - \rho g_y} \cdot C_1^{n+1}$$

non-Newtonian channel flow

$$Effective\ viscosity:$$

$$if\ \dot{\varepsilon}_{II} = A\sigma_{II}^{n}$$

$$\eta_{eff} = \frac{\sigma_{II}}{2\dot{\varepsilon}_{II}} = \frac{\sigma_{xy}}{2\dot{\varepsilon}_{xy}} = \frac{\sigma_{xy}^{1-n}}{2A} = \frac{1}{2A}\left[\left(\frac{\partial P}{\partial y} - \rho g_y\right)\left(x - \frac{L}{2}\right)\right]^{1-n}$$

Exercise:

Calculate the horizontal profiles of the vertical velocity and viscosity across a channel with L = 10 km, (dP/dy – rho*g_y) = 10^4 Pa /m, A = 5*10^{-34}, n = 3, v_{y0} = 0 m/s.

Homework

Read chapter 5 of textbook:

Gerya, T. *Introduction to numerical geodynamic modelling*. Cambridge University Press, 345 pp. (2010)