Numerical Modelling in Geosciences

Lecture 9

Discretization of momentum and continuity equations

Discretization of momentum + continuity equations in 2D

$$Continuity: \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$X - Stokes: \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} - \frac{\partial P}{\partial x} = -\rho g_x$$

$$Y - Stokes: \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} - \frac{\partial P}{\partial y} = -\rho g_y$$

How to apply finite differences

1) Define an Eulerian grid of nodal points

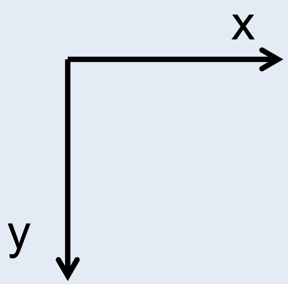
2) Assign input field variables to the nodes

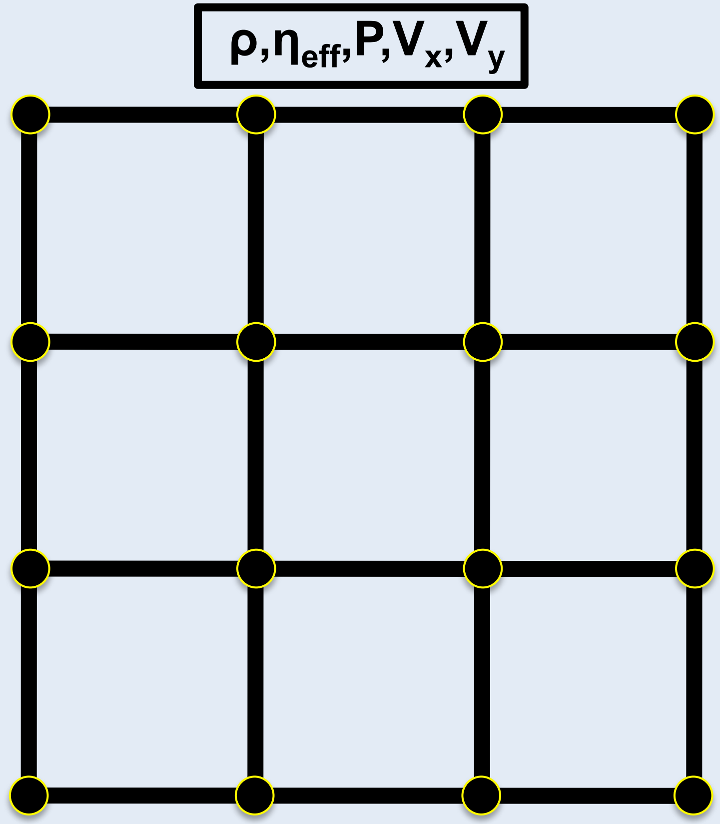
3) Apply PDEs and boundary conditions to each nodal point

4) Solve the system of equations to get output field variables

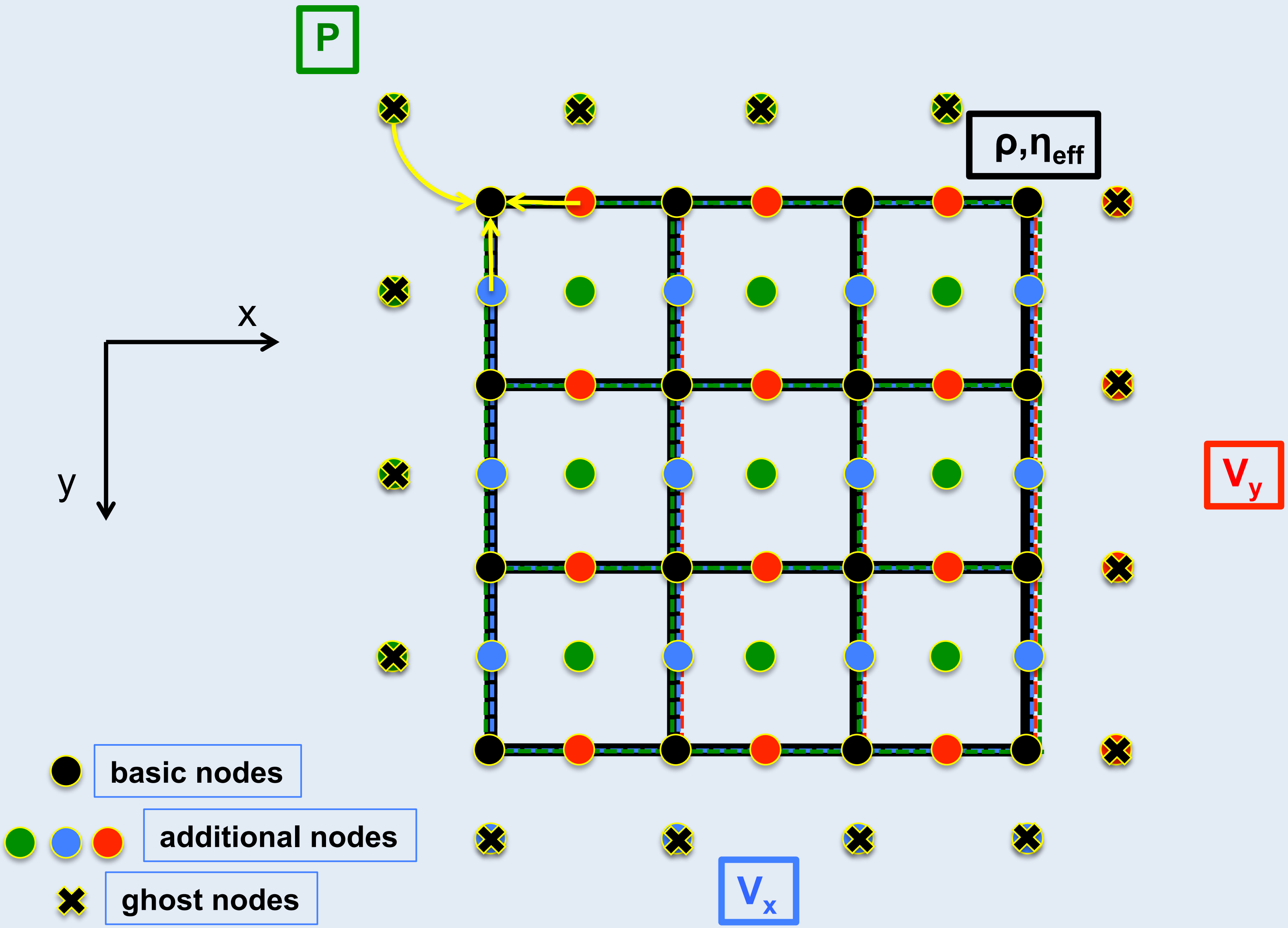
1) Define an Eulerian grid of nodal point

Non-staggered grid





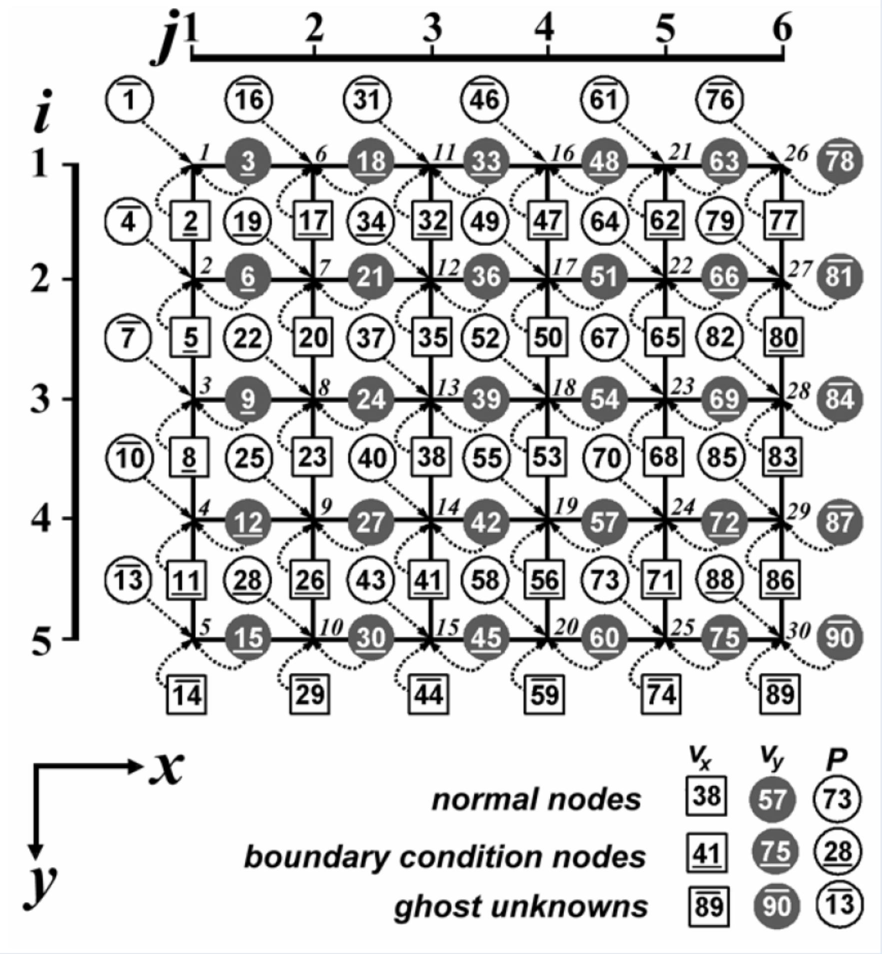
Staggered grid



Indexing

 # equations = # grid points * # of unknown field variables = 90

2D global indexing:
$gi = i + N_i * (j-1)$
$gip = gi*3 - 2$
$gix = gi*3 - 1$
$giy = gi*3$



2) Assign input field variables to the nodes

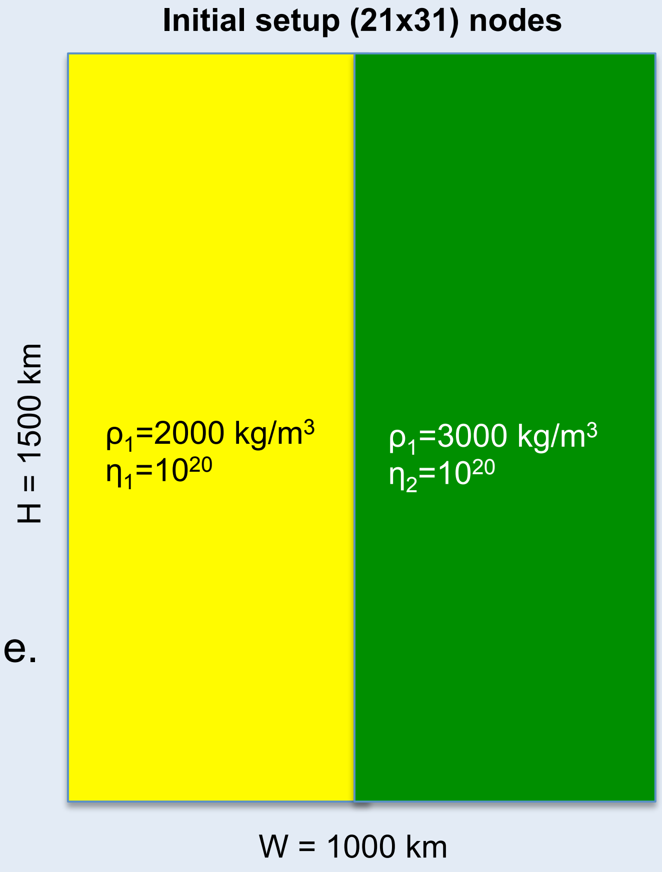
In the case of Stokes equation:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} - \frac{\partial P}{\partial x} = -\rho g_x$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} - \frac{\partial P}{\partial y} = -\rho g_y$$

we need to define as initial input density at each node.
We assume gravity components
and viscosity as constant values for all nodes.

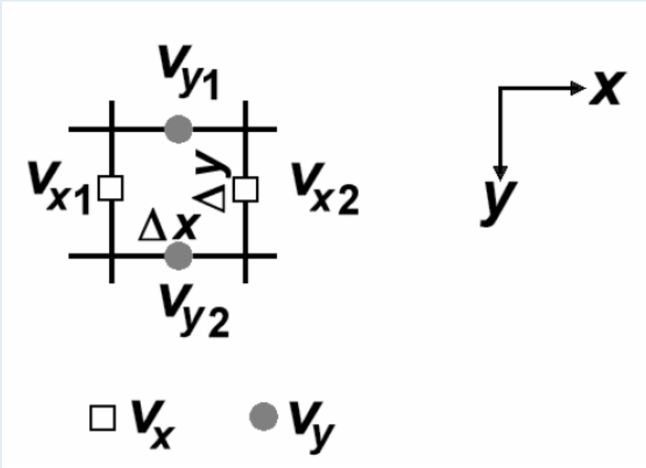


3) Apply PDEs to internal nodes: continuity equation

The continuity equation is formulated in the additional nodes at the centre of the cell where Pressure is obtained.

Coefficients in L matrix must be scaled by 2*η/(Δx+Δy)

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$



3) Apply PDEs to internal nodes: X-Stokes

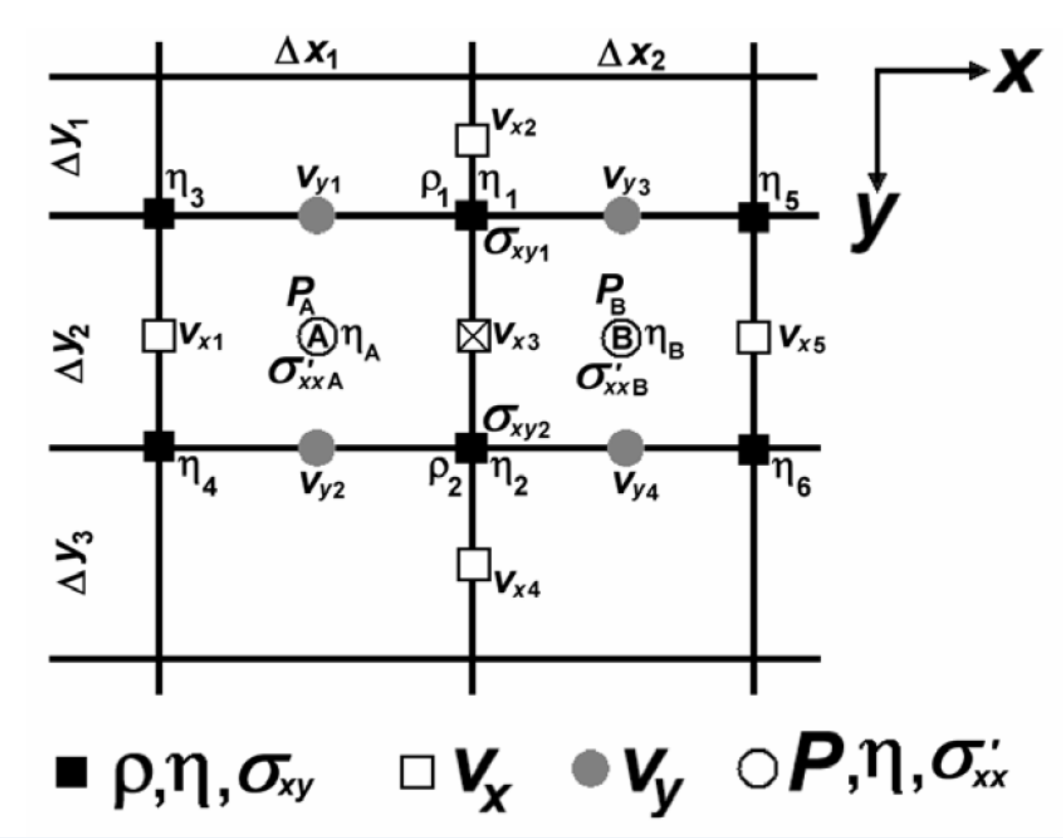
The X-Stokes equation is formulated in the additional nodes halfway two basic nodes along the y direction where Vx is obtained.

Pressure coefficients in L matrix must be scaled by 2*η/(Δx+Δy)

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} - \frac{\partial P}{\partial x} = -\rho g_x$$

$$\eta_A = \frac{\eta_1 + \eta_2 + \eta_3 + \eta_4}{4},$$

$$\eta_B = \frac{\eta_1 + \eta_2 + \eta_5 + \eta_6}{4}.$$



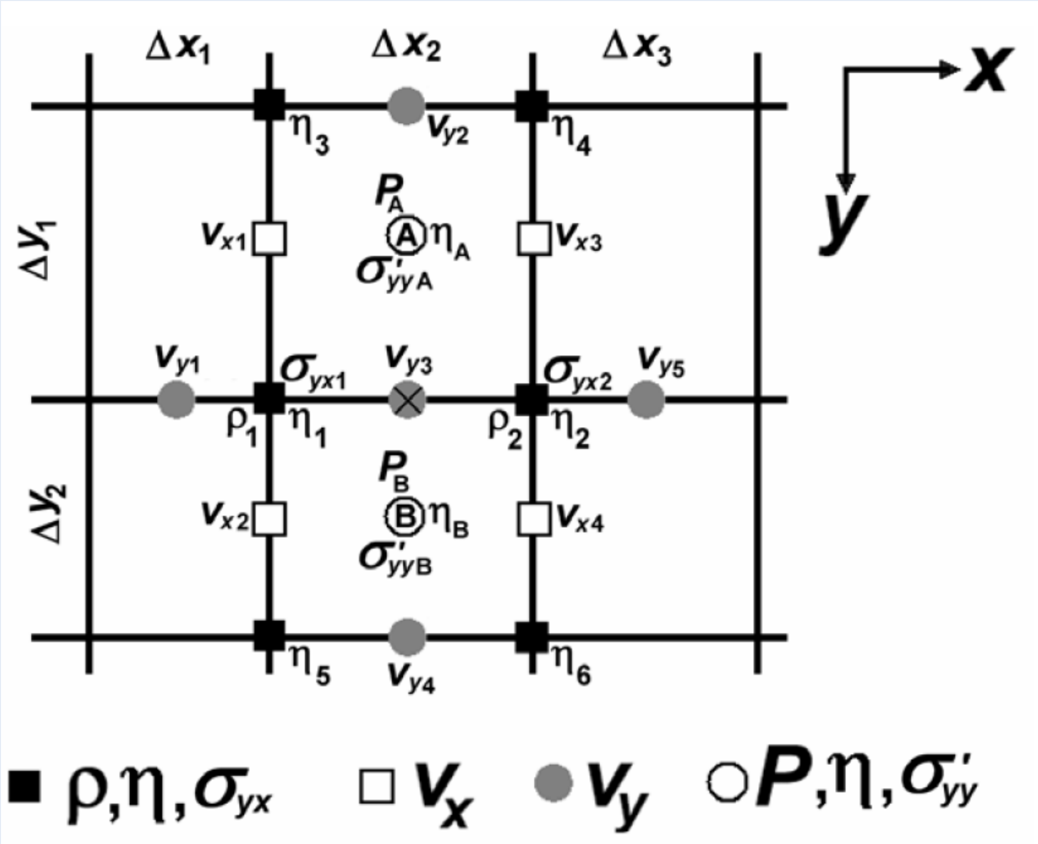
3) Apply PDEs to internal nodes: Y-Stokes

The Y-Stokes equation is formulated in the additional nodes halfway two basic nodes along the x direction where Vy is obtained.
Pressure coefficients in L matrix must be scaled by 2*η/(Δx+Δy)

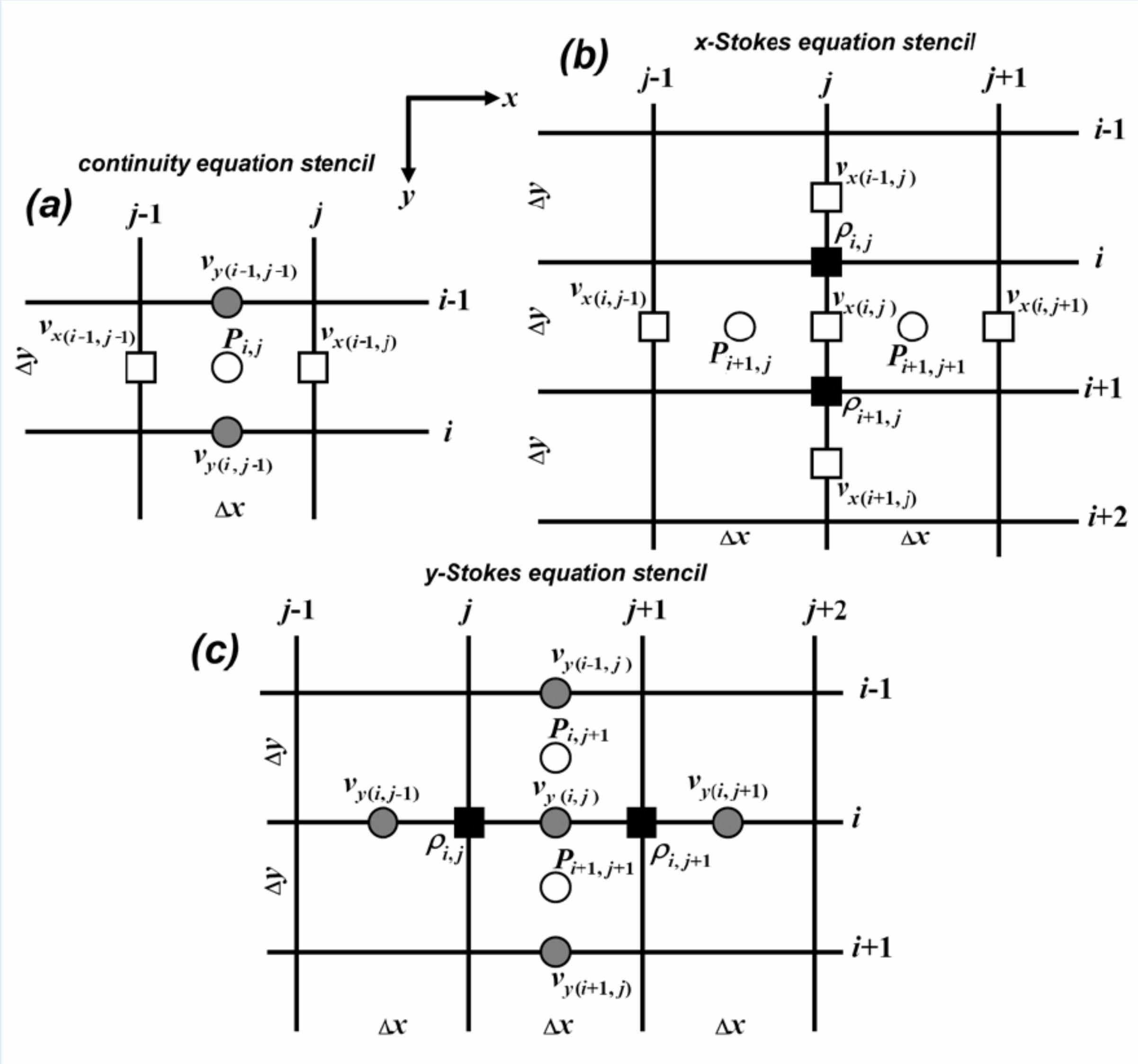
$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} - \frac{\partial P}{\partial y} = -\rho g_y$$

$$\eta_A = \frac{\eta_1 + \eta_2 + \eta_3 + \eta_4}{4},$$

$$\eta_B = \frac{\eta_1 + \eta_2 + \eta_5 + \eta_6}{4}.$$



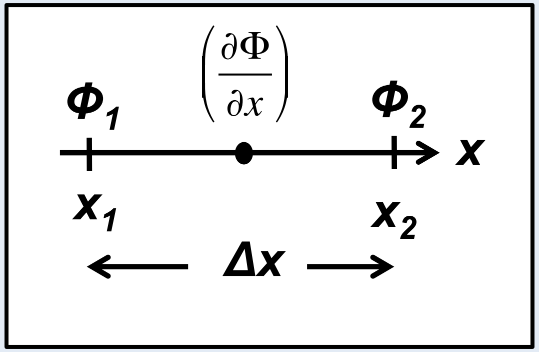
3) Apply PDEs to internal nodes: geometric indexing



Boundary conditions

Dirichlet BC: specify the value of the solution on the boundary nodes

Example: $\Phi_1 = 0$



Neumann BC: specify the value of the derivative of the solution on the boundary nodes

Example: $\frac{\partial \Phi}{\partial x} \approx \frac{\Delta \Phi}{\Delta x} = 4.2 \Rightarrow \Phi_1 = \Phi_2 - 4.2 \cdot \Delta x$

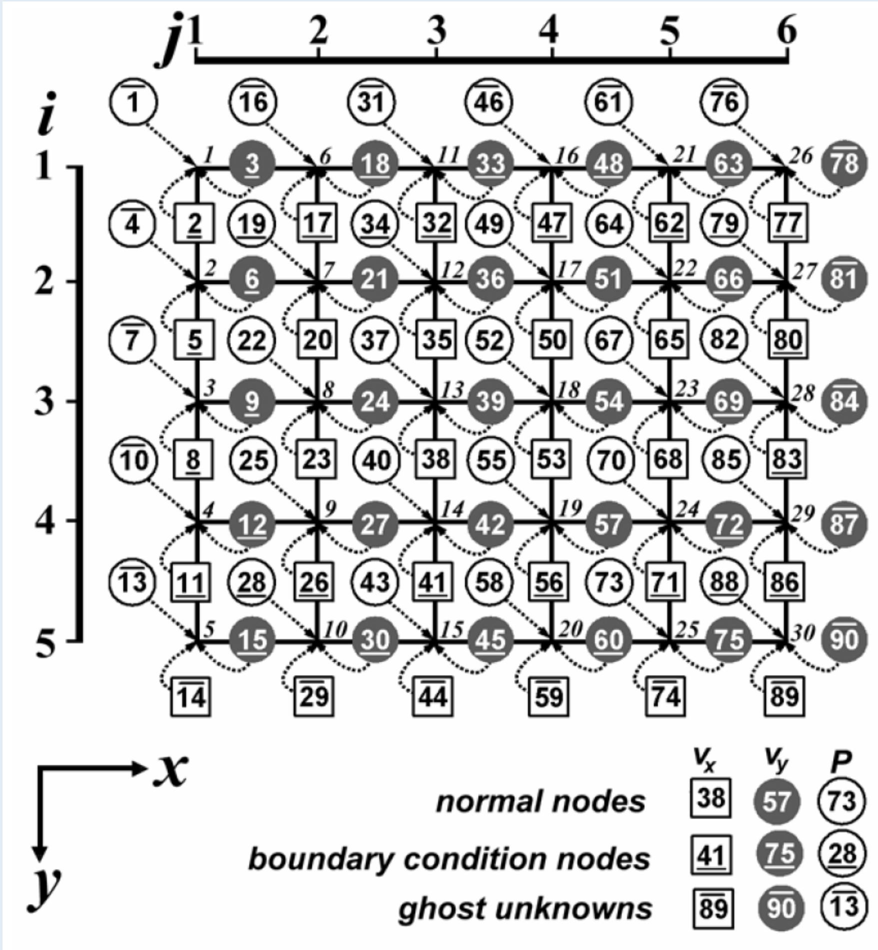
Very important in finite difference method:

For each output (unknown) field variable (e.g., P, T, v_x, v_y, etc.), we must assign a Dirichlet BC to at least 1 node. This is required in order to compute finite differences from an initial value.

Boundary conditions

Left and right coefficients of all BC must be scaled by $4*\eta/(\Delta x+\Delta y)^2$

Set BC at ghost nodes (P = 0, Vx = 0, Vy = 0)

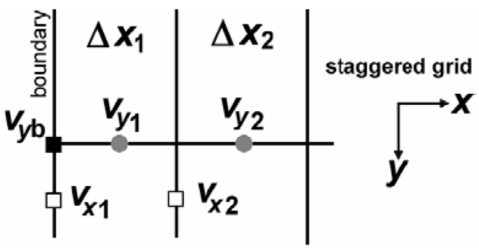


Boundary conditions

Left and right coefficients of all BC must be scaled by $4*\eta/(\Delta x+\Delta y)^2$

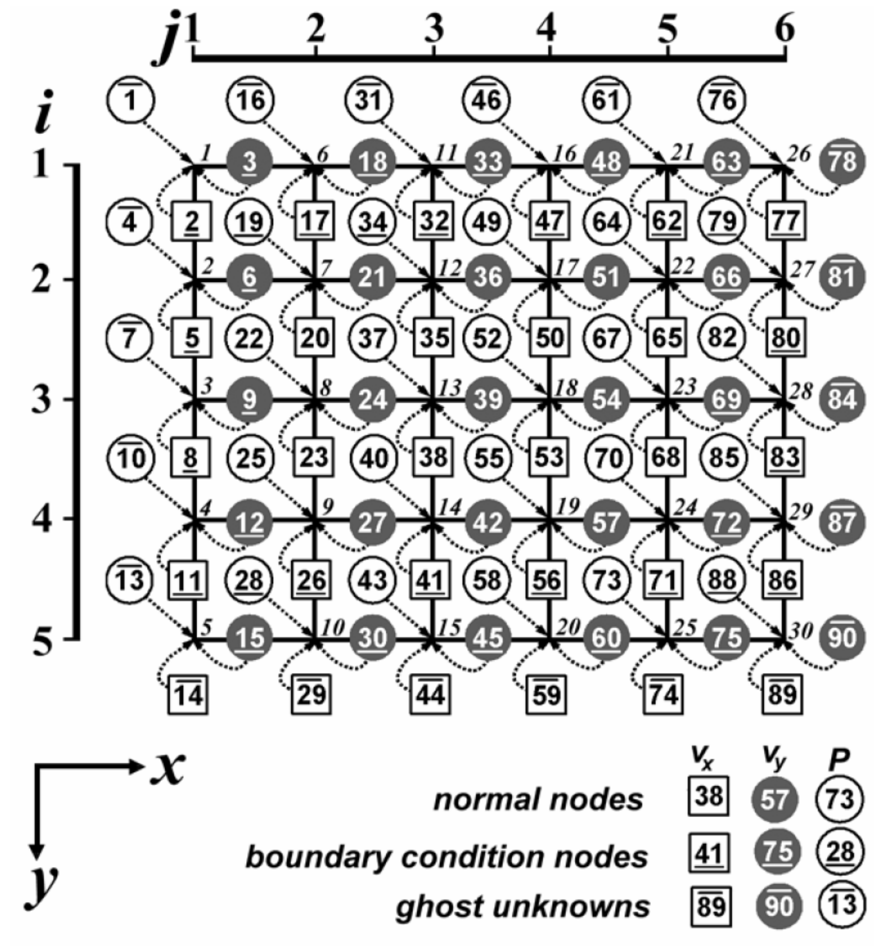
Vx and Vy:

1) free slip
2) no slip
3) moving boundary



$$\text{free slip}, \quad v_{x1}=0, \; v_{y1}=v_{y2},$$

$$\text{no slip}, \quad v_{x1}=0, \; v_{y1}=v_{y2}\frac{\Delta x_1}{2\Delta x_1+\Delta x_2}.$$



How Pressure is obtained from continuity eq.?

Important: global index of pressure must be always bigger than those of surrounding v_x and v_y nodes

We apply Gaussian elimination (Eq. 3.12-3.18) to the analogue system of three equations with variables v_x, v_y and P having the global indexes 1, 2 and 3 respectively

equation A1 (formulated for v_x): $2v_x + 4v_y + 2P = 10$,

equation A2 (formulated for v_y): $3v_x + 9v_y + 6P = 21$,

equation A3 (formulated for P): $v_x + 3v_y = 5$.

Dividing all equations to by a number to normalise the coefficient of v_x , we get

equation B1: $v_x + 2v_y + P = 5$,

equation B2: $v_x + 3v_y + 2P = 7$,

equation B3: $v_x + 3v_y = 5$.

Eliminating v_x by subtracting equation B1 from B2 and B3 yields

equation B1: $v_x + 2v_y + P = 5$,

equation C2: $v_y + P = 2$,

equation C3 : $v_y - P = 0$.

Note that after the elimination operation, P indeed appears in equation C3.

Eliminating v_y by subtracting equation C2 from C3 yields

equation B1: $v_x + 2v_y + P = 5$,

equation C2: $v_y + P = 2$,

equation D3 : $-2P = -2$,

Obtaining the solution for P from equation D3

$$P = -2/(-2) = 1.$$

Obtaining solution for v_y from equation C2

$$v_y = 2 - P = 2 - 1 = 1.$$

Obtaining the solution for v_x from equation B1

$$v_x = 5 - 2v_y - P = 5 - 2 - 1 = 2.$$

If the order is inverted (i.e., the global index of pressure is higher), we cannot solve for pressure

equation 1 (formulated for P): $3v_y + v_x = 5$,

equation 2 (formulated for v_y): $6P + 9v_y + 3v_x = 21$,

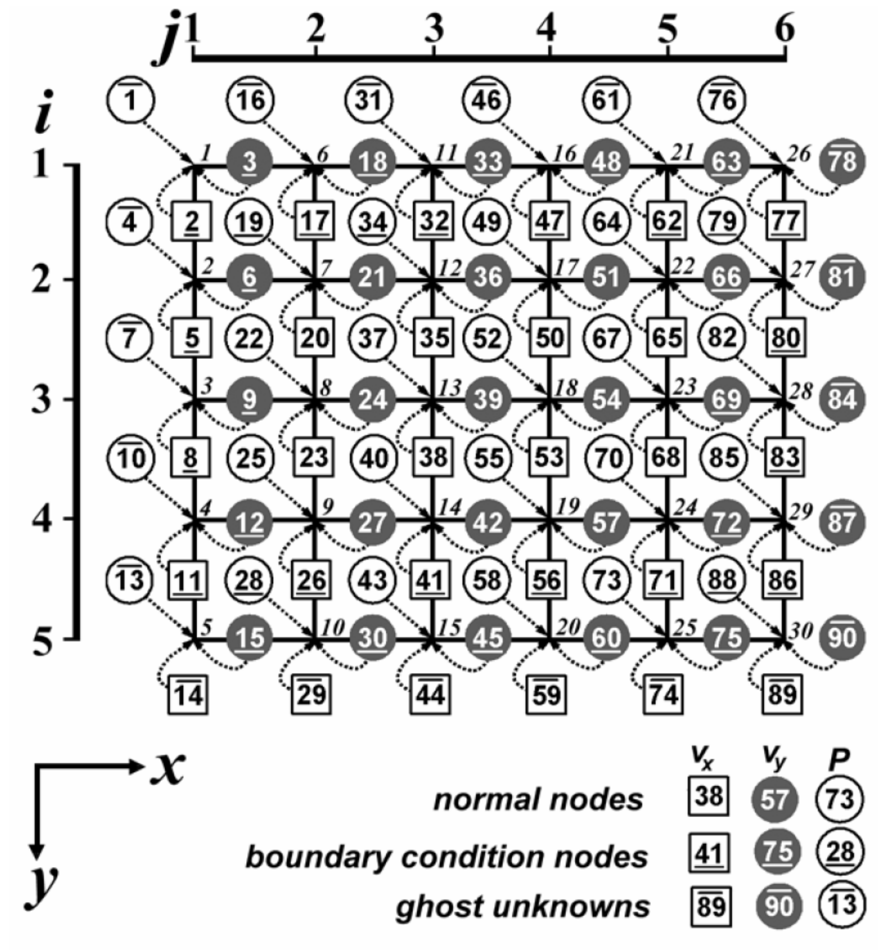
equation 3 (formulated for v_x): $2P + 4v_y + 2v_x = 10$.

Boundary conditions

Left and right coefficients of all BC must be scaled by $4*\eta/(\Delta x+\Delta y)^2$

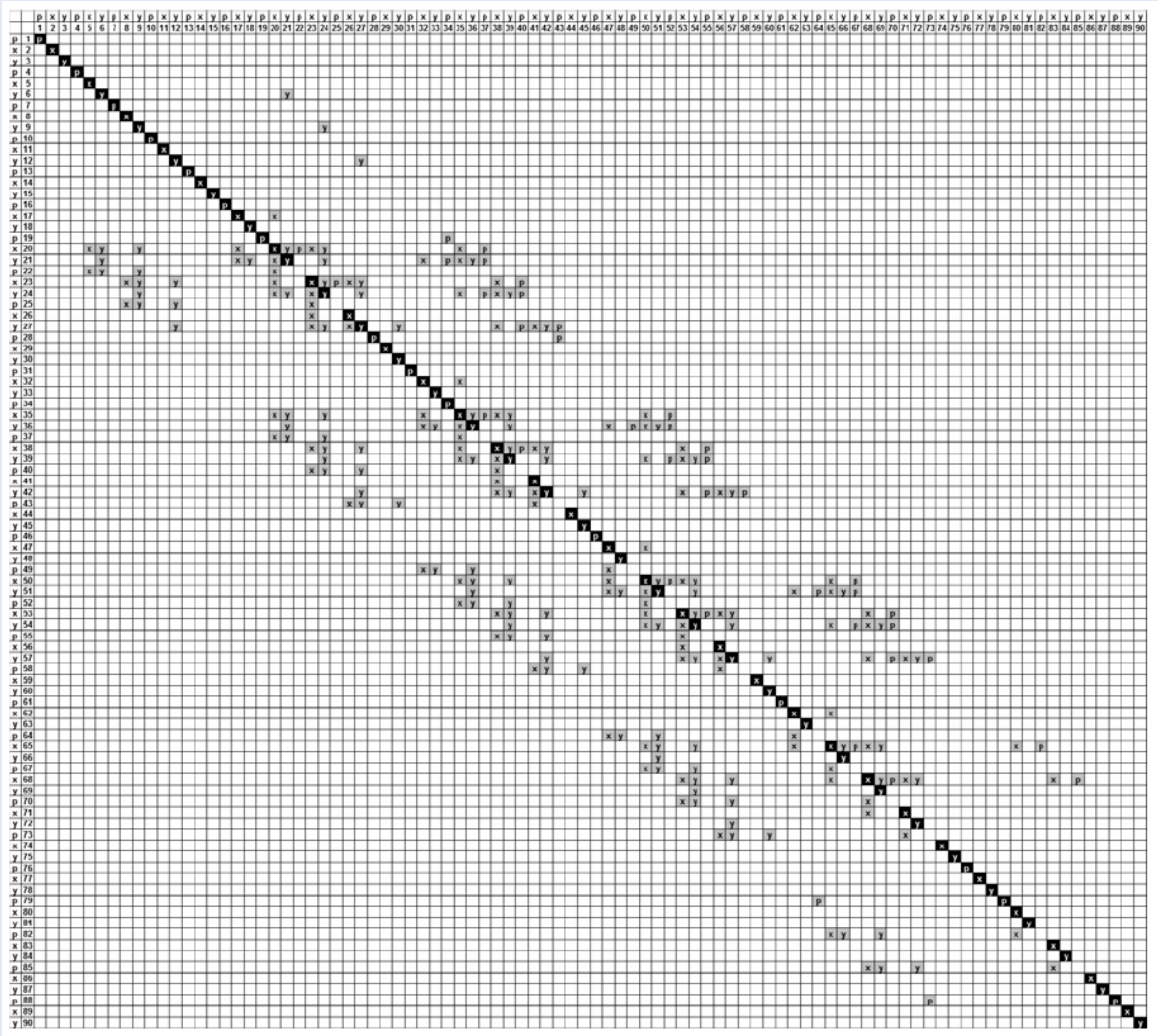
Pressure:
continuity eq. for pressure in a given cell is processed after processing all Stokes eqs. for all surrounding v_x and v_y nodes. Hence, pressure in the 4 cornerns of the grid cannot be computed since these nodes are surrounded by v_x and v_y nodes were Stokes eq. (that contains pressure) has not been formulated, but only BC.

horizontal symmetry condition at the 4 corners (dP/dx =0) + 1 additional node with Dirichlet BC



Left matrix for 6 x 5 grid

Continuity eqs. do not contain coefficient in the diagonal, because P is not defined



Homework

Read chapter 7 of textbook:

Gerya, T. *Introduction to numerical geodynamic modelling*. Cambridge University Press, 345 pp. (2010)